Practice exam 1 problems

1

Write the Taylor series of $f(x) = x^{-2}$ about $a = 1$ as the first 3 terms plus a remainder term.

2

Use Taylor's theorem to find a cubic polynomial $p(x)$ that approximates accurately $f(x) = x^{5/3}$ when x is close to 1.

3

(a) Find the term of order 3 ($i = 3$) in the Taylor series of $\ln(x)$ about $a = 4$.
(b) Write Python code to sum the first 5 terms of this Taylor series for $x = 4.2$

4

Write the Taylor series of $f(x) = \cos(x)$ about $a = \pi$ as the first 3 terms plus a remainder term.

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It's given that $f(a) = b, f'(a) = c, f''(a) = d, f'''(a) = e$; all higher order derivatives of $f(x)$ are zero at $x = a$; and the function f and all its derivatives exist and are continuous between $x = a$ and $x = k$.
(a) If $a = 2, b = 1, c = -2, d = 4, e = 3, k = 5$, what is $h \equiv f(k)$?
(b) Write a Python function to solve for $f(k)$ in general. The first line should be

```
def ffind (a, b, c, d, e, k):
```

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(a) Use 5 iterations of Newton's method to find a root of the function $f(x) = x^5 - 2x + 1$ starting with $x_0 = 0$. (b) Estimate the fractional error of your result.

7

For $R = 27$, apply two iterations of Newton's method with a starting value $x_0 = 5$ to the equation $x^2 - R = 0$ to estimate $\sqrt{R}$.

8

For $R = 30$, apply two iterations of Newton's method with a starting value $x_0 = 3$ to the equation $x^3 - R = 0$ to estimate $\sqrt[3]{R}$.

9

Write a Python script that plots $\log_{10}(x)$ for x between 1 and 100 and labels the x and y axes.

10

Suppose a is some real number that rounds to 100006 and b is another real number that rounds to 99993. Estimate the fractional error in computing $a - b$ if the subtraction is done on a computer with machine epsilon of $\epsilon = 10^{-8}$.

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To 3 significant digits, find the absolute and relative errors of approximating 100 lbf by 442 N (conversion factor: 4.4482216152605 N per lbf). Give correct units.

12

To 3 significant digits, find the absolute and relative errors of approximating 1000 lbm by 453 kg (conversion factor: 0.45359237 kg per lb). Make sure to give correct units.

13

To 3 significant digits, find the absolute and relative errors of approximating $\cos(0.4)$ by the sum of the first 3 nonzero terms of the Maclaurin series of cosine.

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To 5 significant digits, find the absolute and relative errors of approximating the number $\gamma = 0.5772156649\ldots$ by 0.577

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To 4 significant digits, find the absolute and relative errors of approximating the number $\alpha = 0.0072973525664\ldots$ by $\frac{1}{137}$.

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(a) What is decimal 8.5 in base 2?
(b) What is binary 11100 in base 10?

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(a) What is decimal 0.625 in base 2?
(b) What is binary 10111 in base 10?

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If $\mathbf{A} = \mathbf{LU}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 & 1 & 4 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 44 \\ 154 \\ 256 \\ 74 \end{pmatrix},$
find $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$.

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(a) Find the LU decomposition of the matrix $\mathbf{N} = \begin{pmatrix} 2 & -2 & 0 \\ x-2 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$.
(b) For what value of x does $\mathbf{N}$ have no inverse? Explain.

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Given $\mathbf{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 3 & 0 \\ -2 & 1 & 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \mathbf{Ax} = \mathbf{b},$
(a) Find $\mathbf{L}$, $\mathbf{U}$, $\mathbf{P}$ for the LU decomposition of $\mathbf{A}$ with row pivoting.
(b) Find $\mathbf{x}$.

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(a) Use Gauss elimination with row pivoting to find $\mathbf{L}$ and $\mathbf{U}$ factors for the matrix $\mathbf{M} = \begin{pmatrix} 0 & 3 & 4 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(b) How is the product $\mathbf{LU}$ related to $\mathbf{M}$? Explain.

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Suppose the Cholesky factor for a symmetric matrix $\mathbf{A}$ is $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) What is $\mathbf{A}$?

(b) Solve $\mathbf{Ax} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$ for $\mathbf{x}$, using forward and back substitution.

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Let $\mathbf{A}$ be a symmetric positive definite matrix whose Cholesky factor is

$$\mathbf{L} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ -1 & 2 & 1 & 2 \end{pmatrix}.$$

(a) Solve the system $\mathbf{Ax} = \mathbf{b} = \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}^T$. Show and explain all steps.
(b) Find $\mathbf{A}$.

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In each case, determine whether $\mathbf{v}$ is an eigenvector of $\mathbf{A}$. Explain briefly how you know. If it is an eigenvector, state the eigenvalue.

(a) $\mathbf{A} = \begin{pmatrix} -7 & 5 & -2 \\ -14 & 6 & 4 \\ -11 & 5 & 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

(b) $\mathbf{A} = \begin{pmatrix} 11 & 3 & -12 \\ -6 & -4 & 6 \\ 9 & 3 & -10 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(c) $\mathbf{A} = \begin{pmatrix} -1 & 2 & 4 \\ -8 & 1 & -4 \\ 4 & -2 & -1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

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(a) What's the largest (in absolute value) eigenvalue of $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 4 \end{pmatrix}$?

(b) Use 2 iterations of the power method, starting with $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, to estimate the associated eigenvector.

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Use 2 iterations of the power method with initial guess $\mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to estimate the largest eigenvalue of $\mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}$.

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(a) What are the eigenvalues of $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 1 & 5 \end{pmatrix}$?
(b) What are the eigenvectors of $\mathbf{A}$? Scale them so that the first component of each is 1.
(c) If this $\mathbf{A}$ is the coefficient matrix $\mathbf{M}^{-1}\mathbf{K}$ for a system of linear oscillators, what are the periods of the system's modes of oscillation?
(d) Suppose you're given a 10×10 matrix $\mathbf{B}$ that has only positive real eigenvalues. Write Python code to find and display the eigenvector associated with the smallest eigenvalue of $\mathbf{B}$.

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Find the solution $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ to the initial-value problem $\begin{array}{l} x_1'(t) = 8x_1(t) + x_2(t) \\ x_2'(t) = x_1(t) + 8x_2(t) \end{array}$, $\begin{array}{l} x_1(0) = 3 \\ x_2(0) = 2 \end{array}$.

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Find the general solution $x_1(t), x_2(t)$ to the system of second-order differential equations

$$\begin{array}{l} 0 = x_1'' + 2x_1 + x_2 \\ 0 = x_2'' + x_1 + 3x_2. \end{array}$$

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For a two-story shear building with $k_1 = 400, k_2 = 300, m_1 = 4, m_2 = 3$, (a) Write the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$, (b) Write Python code for finding all the oscillation frequencies.

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Solving a shear building problem, you obtain the following result in Python for `(lambda, v) = scipy.linalg.eig(A)`:

```
v =
-2.2801e-01 5.7735e-01 6.5654e-01 -4.2853e-01
-4.2853e-01 5.7735e-01 -2.2801e-01 6.5654e-01
-5.7735e-01 4.7345e-17 -5.7735e-01 -5.7735e-01
-6.5654e-01 -5.7735e-01 4.2853e-01 2.2801e-01
lambda =
1.2061, 10.000, 23.473, 35.320
```

(a) What is the period of the fundamental mode of oscillation?
(b) What is special about the eigenvector associated with this mode?