Practice exam 1 problems with solutions

1

Write the Taylor series of $f(x) = x^{-2}$ about $a = 1$ as the first 3 terms plus a remainder term.

Solution: The general form is $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{6}f'''(\zeta)$ for some ζ between a and x.

So here we can write $f(x) = 1 - 2(x-1) + 3(x-1)^2 - 4\zeta^{-5}(x-1)^3$ for some ζ between 1 and x.

2

Use Taylor's theorem to find a cubic polynomial $p(x)$ that approximates accurately $f(x) = x^{5/3}$ when x is close to 1.

Solution: If $f(x) = x^{5/3}$, $p(x)$ can be obtained from the first four terms of the Taylor series of $f(x)$ about $a = 1$:

$$p(x) = 1 + \frac{5}{3}(x-1) + \frac{5}{9}(x-1)^2 - \frac{5}{81}(x-1)^3.$$

$p(x)$ can also be expanded as $-\frac{5}{81}x^3 + \frac{20}{27}x^2 + \frac{10}{27}x - \frac{4}{81}$.

3

(a) Find the term of order 3 ($i = 3$) in the Taylor series of $\ln(x)$ about $a = 4$.
(b) Write Python code to sum the first 5 terms of this Taylor series for $x = 4.2$

Solution: (a) The order-i derivative ($i > 0$) of $\log(x)$ is $(-1)^{i-1}(i-1)!x^{-i}$. The Taylor series of $\log(x)$ about a is therefore

$$\log(a) + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{ia^i}(x-a)^i.$$

Thus, if $a = 4$, the $i = 3$ term is $(x-4)^3/192$.
(b)

```python
from math import log
a = 4
x = 4.2
n = 5
series_sum = log(a)
for i in range(1,n):
series_sum = series_sum + (x - a)**i * (-1)**(i-1) / (i * a**i)
```

4

Write the Taylor series of $f(x) = \cos(x)$ about $a = \pi$ as the first 3 terms plus a remainder term.

Solution: The general form is $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{6}f'''(\zeta)$ for some ζ between a and x.

So here we can write $f(x) = -1 + \frac{1}{2}(x-\pi)^2 + \frac{1}{6}\sin(\zeta)(x-\pi)^3$ for some ζ between π and x.

5

It's given that $f(a) = b, f'(a) = c, f''(a) = d, f'''(a) = e$; all higher order derivatives of $f(x)$ are zero at $x = a$; and the function f and all its derivatives exist and are continuous between $x = a$ and $x = k$.
(a) If $a = 2, b = 1, c = -2, d = 4, e = 3, k = 5$, what is $h \equiv f(k)$?
(b) Write a Python function to solve for $f(k)$ in general. The first line should be

```python
def ffind (a, b, c, d, e, k):
```

Solution: Based on Taylor's theorem, the general formula would be $h = b + cn + dn^2/2 + en^3/6$, where $n \equiv k - a$.
(a) Substituting the given values, $h = 1 + (-2)\cdot 3 + 4 \cdot 3^2/2 + 3 \cdot 3^3/6 = \mathbf{26.5}$
(b)

```python
def ffind (a, b, c, d, e, k):
        n = k - a
        h = b + c*n + d*n**2/2 + e*n**3/6
        return h
```

6

(a) Use 5 iterations of Newton's method to find a root of the function $f(x) = x^5 - 2x + 1$ starting with $x_0 = 0$. (b) Estimate the fractional error of your result.

Solution: (a) If $f(x) = x^5 - 2x + 1$, $f'(x) = 5x^4 - 2$, and successive iterations give
0

0.5
$0.\overline{518}$
0.518790000907635
0.518790063675881
0.518790063675884
(b) Using the difference between the last two iterations to estimate the error, we get $\frac{|x_i - x_{i-1}|}{x_i} = 6 \times 10^{-15}$. Note though that even if the last two values were floating-point numbers that were identical at the precision of our computer or calculator, we should not assume that the actual fractional error is smaller than machine epsilon, or (assuming double precision) $\sim 10^{-16}$.

7

For $R = 27$, apply two iterations of Newton's method with a starting value $x_0 = 5$ to the equation $x^2 - R = 0$ to estimate $\sqrt{R}$.

Solution: $x_{i+1} \leftarrow x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - R}{2x_i} = \frac{x_i}{2} + \frac{R}{2x_i} = \frac{x_i + R/x_i}{2}$
So for $R = 27$, $x_0 = 5, x_1 = 5.2, x_2 = \mathbf{5.1962}$

8

For $R = 30$, apply two iterations of Newton's method with a starting value $x_0 = 3$ to the equation $x^3 - R = 0$ to estimate $\sqrt[3]{R}$.

Solution: $x_{i+1} \leftarrow x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - R}{3x_i^2} = \frac{2x_i}{3} + \frac{R}{3x_i^2} = \frac{2x_i + R/x_i^2}{3}$
So $x_0 = 3, x_1 = 3\frac{1}{9}, x_2 = \mathbf{3.1072}$

9

Write a Python script that plots $\log_{10}(x)$ for x between 1 and 100 and labels the x and y axes.

Solution:
```
from numpy import log10, linspace
from matplotlib.pyplot import plot, xlabel, ylabel
x = linspace(1, 100, 1000) #generate enough points for a smooth curve
y = log10(x)
plot(x, y)
xlabel('x'), ylabel('y')
```

10

Suppose a is some real number that rounds to 100006 and b is another real number that rounds to 99993. Estimate the fractional error in computing $a - b$ if the subtraction is done on a computer with machine epsilon of $\epsilon = 10^{-8}$.

Solution: The absolute roundoff error for $a - b$ can be as high as around $\epsilon(|a| + |b|)$. The fractional error is therefore bounded by $\epsilon\frac{|a|+|b|}{|a-b|}$, where the rounded values can be used to estimate $|a|, |b|, |a-b|$. This gives $10^{-8}\frac{|100006|+|99993|}{|100006-99993|}$, or around 1.5×10^{-4}.

11

To 3 significant digits, find the absolute and relative errors of approximating 100 lbf by 442 N (conversion factor: 4.4482216152605 N per lbf). Give correct units.

Solution: Absolute error: $|100 - 442/4.4482216152605| = \mathbf{0.634}$ **lbf** (or 2.82 N)

Relative error: $0.634/|100| = \mathbf{6.34 \times 10^{-3}}$ (dimensionless)

12

To 3 significant digits, find the absolute and relative errors of approximating 1000 lbm by 453 kg (conversion factor: 0.45359237 kg per lb). Make sure to give correct units.

Solution: Absolute error: $|1000 - 453/0.45359237| = \mathbf{1.31}$ **lb** (or 0.592 kg)

Relative error: $1.31/|1000| = \mathbf{1.31 \times 10^{-3}}$ (dimensionless)

13

To 3 significant digits, find the absolute and relative errors of approximating $\cos(0.4)$ by the sum of the first 3 nonzero terms of the Maclaurin series of cosine.

Solution: The Taylor series approximation is $1 - (0.4)^2/2 + (0.4)^4/24$. Compared to a more accurate value of the cosine, it has absolute error 5.67×10^{-6} and relative error 6.16×10^{-6}.

14

To 5 significant digits, find the absolute and relative errors of approximating the number $\gamma = 0.5772156649\ldots$ by 0.577

Solution: Absolute error: 2.1566×10^{-4}; relative error: 3.7363×10^{-4}.

15

To 4 significant digits, find the absolute and relative errors of approximating the number $\alpha = 0.0072973525664\ldots$ by $\frac{1}{137}$.

Solution: Absolute error: 1.918×10^{-6}; relative error: 2.628×10^{-4}.

16

(a) What is decimal 8.5 in base 2?
(b) What is binary 11100 in base 10?

Solution: (a) 8.5 is 2^3 plus 1/2, or binary 1000.1
(b) Binary 11100 is $4 + 8 + 16$, or 28.

17

(a) What is decimal 0.625 in base 2?
(b) What is binary 10111 in base 10?

Solution: (a) 0.625 is 1/2 plus 1/8, or binary 0.101
(b) Binary 10111 is $1 + 2 + 4 + 16$, or 23.

18

If $\mathbf{A} = \mathbf{LU}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 & 1 & 4 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 44 \\ 154 \\ 256 \\ 74 \end{pmatrix}$, find $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$.

Solution: Given the LU factorization of $\mathbf{A}$, we solve $\mathbf{Ly} = \mathbf{b}$ by forward substitution to get $\mathbf{y} = \begin{pmatrix} 44 \\ 22 \\ 14 \\ 16 \end{pmatrix}$, followed by solving $\mathbf{Ux} = \mathbf{y}$ by back substitution to get $\mathbf{x} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 4 \end{pmatrix}$.

19

(a) Find the LU decomposition of the matrix $\mathbf{N} = \begin{pmatrix} 2 & -2 & 0 \\ x-2 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$.
(b) For what value of x does $\mathbf{N}$ have no inverse? Explain.

Solution: (a) The factors are $\mathbf{U} = \begin{pmatrix} 2 & -2 & 0 \\ 0 & x & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{x}{2}-1 & 1 & 0 \\ 0 & -\frac{1}{x} & 1 \end{pmatrix}$.
(b) The matrix determinant is $6x$, which is zero only when $x = 0$, indicating that the matrix has no inverse then (which can also be seen from the second row being a multiple of the first, and thus the rows are not linearly independent).

20

Given $\mathbf{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 3 & 0 \\ -2 & 1 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{Ax} = \mathbf{b}$,
(a) Find $\mathbf{L}$, $\mathbf{U}$, $\mathbf{P}$ for the LU decomposition of $\mathbf{A}$ with row pivoting.
(b) Find $\mathbf{x}$.

Solution: (a) Using Gauss elimination with row pivoting, the steps are

$$\begin{pmatrix} 3 & -2 & 1 & (1) \\ 0 & 3 & 0 & (2) \\ -2 & 1 & 0 & (3) \end{pmatrix} \to \begin{pmatrix} 3 & -2 & 1 & (1) \\ 0| & 3 & 0 & (2) \\ -\frac{2}{3}| & -\frac{1}{3} & \frac{2}{3} & (3) \end{pmatrix} \to \begin{pmatrix} 3 & -2 & 1 & (1) \\ 0| & 3 & 0 & (2) \\ -\frac{2}{3} & -\frac{1}{9}| & \frac{2}{3} & (3) \end{pmatrix}$$

No row pivoting is actually done in this case.
The factors are $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{3} & -\frac{1}{9} & 1 \end{pmatrix}$, $\mathbf{U} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Given the LU factorization of $\mathbf{A}$, we solve $\mathbf{Ly} = \mathbf{Pb}$ by forward substitution to get $\mathbf{y} = \begin{pmatrix} 2 \\ 0 \\ 4/3 \end{pmatrix}$, followed by solving $\mathbf{Ux} = \mathbf{y}$ by back substitution to get $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

21

(a) Use Gauss elimination with row pivoting to find $\mathbf{L}$ and $\mathbf{U}$ factors for the matrix $\mathbf{M} = \begin{pmatrix} 0 & 3 & 4 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(b) How is the product $\mathbf{LU}$ related to $\mathbf{M}$? Explain.

Solution: (a) Row pivoting in this case interchanges the first and second rows of $\mathbf{M}$.

The factors are $\mathbf{U} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 4 & 4 \\ 0 & 0 & -2/3 & -5/3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$, $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) The product $\mathbf{LU}$ is the original matrix $\mathbf{M}$, but with the rows interchanged in the same sequence followed in the row pivoting steps. So $\mathbf{LU} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 4 & 4 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

22

Suppose the Cholesky factor for a symmetric matrix $\mathbf{A}$ is $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) What is $\mathbf{A}$?

(b) Solve $\mathbf{Ax} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$ for $\mathbf{x}$, using forward and back substitution.

Solution: (a) By definition of the Cholesky factorization, we have $\mathbf{A} = \mathbf{LL}^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(b) Since $\mathbf{LL}^T\mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$, forward substitution on $\mathbf{Ly} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$ gives $\mathbf{y} = \begin{pmatrix} 4 \\ -1/2 \\ 3 \end{pmatrix}$, followed by back substitution on $\mathbf{L}^T\mathbf{x} = \mathbf{y}$ to get $\mathbf{x} = \begin{pmatrix} 5/4 \\ -1/4 \\ 3 \end{pmatrix}$.

23

Let $\mathbf{A}$ be a symmetric positive definite matrix whose Cholesky factor is

$$\mathbf{L} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ -1 & 2 & 1 & 2 \end{pmatrix}.$$

(a) Solve the system $\mathbf{Ax} = \mathbf{b} = \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}^T$. Show and explain all steps.
(b) Find $\mathbf{A}$.

Solution: (a) Solve the lower triangular system $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$ using forward substitution, then solve the upper triangular system $\mathbf{L}^T\mathbf{x} = \mathbf{y}$ using back substitution. (b) Matrix multiply (by definition, $\mathbf{A} = \mathbf{LL}^T$).
In this case, we get $\mathbf{y} = \begin{pmatrix} 0 & 1 & 2/3 & -4/3 \end{pmatrix}^T$, $\mathbf{x} = \begin{pmatrix} 1/6 & 17/9 & 4/9 & -2/3 \end{pmatrix}^T$, and

$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 4 & -2 \\ -2 & 2 & -1 & 3 \\ 4 & -1 & 14 & 3 \\ -2 & 3 & 3 & 10 \end{pmatrix}.$$

24

In each case, determine whether $\mathbf{v}$ is an eigenvector of $\mathbf{A}$. Explain briefly how you know. If it is an eigenvector, state the eigenvalue.

(a) $\mathbf{A} = \begin{pmatrix} -7 & 5 & -2 \\ -14 & 6 & 4 \\ -11 & 5 & 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$

(b) $\mathbf{A} = \begin{pmatrix} 11 & 3 & -12 \\ -6 & -4 & 6 \\ 9 & 3 & -10 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$

(c) $\mathbf{A} = \begin{pmatrix} -1 & 2 & 4 \\ -8 & 1 & -4 \\ 4 & -2 & -1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$

Solution: In each case, you need to compute the matrix product $\mathbf{Av}$. If it is a multiple of $\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ and the multiplier is the eigenvalue.
Here, in (a) and (b) $\mathbf{v}$ isn't an eigenvector, whereas in (c), $\mathbf{v}$ is an eigenvector with an eigenvalue of -1.

25

(a) What's the largest (in absolute value) eigenvalue of $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 4 \end{pmatrix}$?

(b) Use 2 iterations of the power method, starting with $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, to estimate the associated eigenvector.

Solution: (a) Since $\mathbf{B}$ is triangular, the elements on its main diagonal are its eigenvalues. So the largest eigenvalue is **5**.

(b) Dividing at each step by the largest absolute values in $\mathbf{v}$ (the infinity norm), we have $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2/3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{8/17} \end{pmatrix}$.

26

Use 2 iterations of the power method with initial guess $\mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to estimate the largest eigenvalue of $\mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}$.

Solution: $\lambda_1 = ||\mathbf{A}\mathbf{v}_0||_2 = \sqrt{17} = 4.123$

$\mathbf{v}_1 = \mathbf{A}\mathbf{v}_0/\lambda_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} /\sqrt{17}$

$\lambda_2 = ||\mathbf{A}\mathbf{v}_1||_2 = \sqrt{\frac{293}{17}} = \mathbf{4.152}$.

27

(a) What are the eigenvalues of $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 1 & 5 \end{pmatrix}$?

(b) What are the eigenvectors of $\mathbf{A}$? Scale them so that the first component of each is 1.

(c) If this $\mathbf{A}$ is the coefficient matrix $\mathbf{M}^{-1}\mathbf{K}$ for a system of linear oscillators, what are the periods of the system's modes of oscillation?

(d) Suppose you're given a 10×10 matrix $\mathbf{B}$ that has only positive real eigenvalues. Write Python code to find and display the eigenvector associated with the smallest eigenvalue of $\mathbf{B}$.

Solution: (a) For $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 1 & 5 \end{pmatrix}$, the characteristic polynomial is $(6 - \lambda)(5-\lambda) - 1$, with roots at $\frac{11 \pm \sqrt{121-116}}{2}$, or $\frac{11\pm\sqrt{5}}{2}$.

(b) They are $(1, \frac{-1\pm\sqrt{5}}{2})$.

(c) 2π divided by the eigenvalue square roots, or 2.4424 and 3.0015

(d)

```
from scipy.linalg import eig
from numpy import argmin (d, v) = eig(B)
v[:, argmin(d)]
```

28

Find the solution $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ to the initial-value problem $\begin{array}{c} x_1'(t) = 8x_1(t) + x_2(t) \\ x_2'(t) = x_1(t) + 8x_2(t) \end{array}$, $\begin{array}{c} x_1(0) = 3 \\ x_2(0) = 2 \end{array}$.

Solution: We can write the system as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with coefficient matrix $\mathbf{A} = \begin{pmatrix} 8 & 1 \\ 1 & 8 \end{pmatrix}$, which has eigenvalues $\lambda = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$ and eigenvectors $\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. The general solution is therefore $\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{9t}$. The initial conditions at $t = 0$ require $c_1 = -\frac{1}{2}, c_2 = \frac{5}{2}$.

29

Find the general solution $x_1(t), x_2(t)$ to the system of second-order differential equations

$$\begin{array}{c} 0 = x_1'' + 2x_1 + x_2 \\ 0 = x_2'' + x_1 + 3x_2. \end{array}$$

Solution: This can be written in matrix form as $\mathbf{x}'' = -\mathbf{A}\mathbf{x}$, with $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. As we saw in class, the general solution is $\mathbf{x}(t) = c_1 \sin(\sqrt{\lambda_1} t)\mathbf{v}_1 + d_1 \cos(\sqrt{\lambda_1} t)\mathbf{v}_1 + c_2 \sin(\sqrt{\lambda_2} t)\mathbf{v}_2 + d_2 \cos(\sqrt{\lambda_2} t)\mathbf{v}_2$, where $\lambda_i, \mathbf{v}_i$ are the eigenvalues and corresponding eigenvectors of $\mathbf{A}$ and the c_i and d_i can be any numbers. For this case the eigenvalues are $2.5 \pm \sqrt{1.25}$ and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0.5 \pm \sqrt{1.25} \end{pmatrix}$ (or any nonzero multiples of those vectors).

30

For a two-story shear building with $k_1 = 400, k_2 = 300, m_1 = 4, m_2 = 3$, (a) Write the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$, (b) Write Python code for finding all the oscillation frequencies.

Solution: (a) $\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

$\mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} = \begin{pmatrix} 700 & -300 \\ -300 & 300 \end{pmatrix}$

(b)

```
from numpy import array, diag, sqrt, pi
from numpy.linalg import solve
from scipy.linalg import eig
m1 = 4; m2 = 3;
```

```
k1 = 400; k2 = 300;
M = diag([m1, m2]); K = array([[k1+k2, -k2], [-k2, k2]])
A = solve(M, K)
(l, v) = eig(A) # vector with all eigenvalues
f = sqrt(l) ./ (2*pi) #oscillation frequencies (cycles per unit time;
periods are 1 / f)
```

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Solving a shear building problem, you obtain the following result in Python for `(lambda, v) = scipy.linalg.eig(A)`:

```
v =
-2.2801e-01 5.7735e-01 6.5654e-01 -4.2853e-01
-4.2853e-01 5.7735e-01 -2.2801e-01 6.5654e-01
-5.7735e-01 4.7345e-17 -5.7735e-01 -5.7735e-01
-6.5654e-01 -5.7735e-01 4.2853e-01 2.2801e-01
lambda =
1.2061, 10.000, 23.473, 35.320
```

(a) What is the period of the fundamental mode of oscillation?
(b) What is special about the eigenvector associated with this mode?

Solution: (a) 2π divided by the square root of the smallest eigenvalue, or **5.721** time units
(b) Since the smallest eigenvalue is the first one, the associated eigenvector is given by the first column of `v`. It is the only eigenvector which has all its elements the same sign. This corresponds to an oscillation where the whole building sways to each side together, with the higher stories swaying the farthest.