Practice exam 2 problems

1

Approximate the first two derivatives of $y(x)$ at the given points using second order accurate finite differences (centered if possible).

x	-1	0	1	2
y	2	2	3	4
y'	-	-	-	-
y''	-	-	-	-

2

Given the following data, approximate the first four derivatives of $f(x)$ at $x = 1$ using centered finite differences.

x	-1	0	1	2	3
$f(x)$	1	2	4	3	2

3

(a) Find D_0^2 for Richardson extrapolation to estimate the derivative of $f(x)$ at $x = 1.1$, with initial step size $h_0 = 0.4$. Use the function values given below, and show at least 7 significant figures.
(b) What is the estimated fractional error in this obtained $f'(x)$?

x	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$f(x)$	61255	67322	74199	81907	90488	100000	110513	122106	134870	148902	164311

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(a) Find D_0^2 for Richardson extrapolation to estimate the derivative of $f(x)$ at $x = 1$, with initial step size $\Delta_0 = 0.4$. Use the function values given below.
(b) What is the estimated relative error in this estimated $f'(x)$?

x	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$f(x)$	70711	73602	77906	83651	90953	100000	111053	124456	140646	160169	183712

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Consider the composite trapezoid rule approximation T_{78} of $\int_0^1 x^{95}dx$.

(a) Does T_{78} overestimate or underestimate the exact value? Explain how you know.

(b) Find a bound for the absolute error in T_{78} using the result that $\text{Error}(T_N) \leq \frac{M(b-a)^3}{12N^2}$, where M is the least upper bound for all absolute values of the second derivatives over $[a, b]$ of the function being integrated.

6

(a) Estimate the integral $\int_0^2 x^{10}dx$ using the simple Simpson's rule.
(b) Find the minimum number N of equal-width subdivisions needed to guarantee that the absolute error in the composite Simpson rule estimate for this integral is under 10^{-7} using the error bound $\frac{K_4(b-a)^5}{2880N^4}$.

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Estimate $I = \int_2^4 y(x)dx$ given the values below (a) using the simple Simpson rule,
(b) using the composite Simpson rule with $n = 2$ equal-length intervals.
(c) Based on (a) and (b), estimate what n would be needed for the absolute error in estimating I using the composite Simpson rule to be under 10^{-3}.

x	2	2.5	3	3.5	4
$y(x)$	100	67	50	40	33

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Estimate the integral of $e^{(x^2)}$ between $a = 0$ and $b = 1$ using Simpson's rule with
(a) one subinterval,
(b) two subintervals.
(c) Estimate your fractional error based on the difference between the two results.

9

An irregular lot is 100 ft long, and its width as measured at various points is as follows:

position (ft)	0	25	50	75	100
width (ft)	10	15	20	23	25.

Estimate the lot's area as accurately as possible using the composite Simpson rule.

10

Starting from standstill at $t = 0$, the acceleration of an elevator is measured as

t(s)	0	5	10	15	20	23	30
a(m/s^2)	0	2.2	3.6	5.2	6.0	-4.0	-8.1

(a) Estimate the velocity at $t = 20$ using the composite trapezoid rule.
(b) Write a Python script to estimate the velocity at $t = 20$ by fitting an interpolating polynomial to the data and integrating it.

11

Write Python code to estimate the probability of falling between $x = 1$ and $x = 2$ for the probability density function $p(x) = e^{-(e^{-x}+x)}$.

12

Calculate the Romberg integration estimator R_0^2 of

$$\int_0^1 \frac{x^3}{x+1}\,dx.$$

13

(a) Use Romberg integration with $j_{\max} = 2$ to estimate

$$I = \int_0^2 \frac{x}{x+3}dx.$$

(b) Estimate the relative error based on the difference between your two most accurate estimates.

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(a) Fill in the first 3 levels of Romberg integration to estimate the integral of $\sqrt{x}$ for x between 0 and 3.
(b) Estimate the absolute error based on the difference between the two most accurate estimates above.
(c) Find the actual absolute error of the most accurate estimate above compared to the analytic integral.

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(a) Use Romberg integration with $j_{\max} = 3$ to estimate $I = \int_0^1 (e^x - 4x)dx$.
(b) Find the solution analytically, and based on this calculate the fractional error of part a.

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Use Romberg integration with $j_{\max} = 2$ to estimate $I = \int_0^2 \frac{x^2}{x^2+3}dx$.

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Use the (explicit) Euler method to solve for $y(x = 1)$ if $y(x = 0) = 2$ and $y' = \frac{y}{x^2+2}$.

(a) With step size $h = 1$.
(b) With step size $h = 0.5$.
(c) Explain how you can estimate the fractional error in your solution using the above results.

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Estimate $y(5)$ for the initial-value problem

$$\frac{dy}{dt} = \sqrt{y} - t + 1, \;\; y(1) = 3$$

using Euler's method (RK1) with step size $h = 1$.

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(a) Use Euler's method with step size $h = 1/4$ to estimate $y(1)$ given $\frac{dy}{dt} = -2y, y(0) = 3$.
(b) Find the absolute error of this estimate compared to the analytic solution $y(t) = 3e^{-2t}$.
(c) Write Python code to solve for and display $y(1)$ using `ode45`.

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(a) Use Euler's method with a step size h of $1/3$ to estimate $y(t = 1)$ if $y(t = 0) = 0, dy/dt = y + 2\sqrt{t}$.
(b) Estimate $y(t = 1)$ for the same problem with the RK4 method and $h = 1$.
(c) Write Python code that uses `scipy.integrate.solve_ivp` to estimate $y(t = 1)$ and displays the estimated value.

21

Let $y(t)$ be the solution to $y' = 4te^{-y}$ satisfying $y(0) = 3$. Use Euler's Method with time step $h = 0.1$ to approximate $y(0.1), y(0.2), \ldots, y(0.5)$.

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(a) Use Euler's method with a step size h of $1/3$ to estimate $y(t = 1)$ if $y(t = 0) = 0, dy/dt = y + 4t^2$.
(b) Estimate $y(t = 1)$ for the same problem with the RK4 method and $h = 1$.

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Estimate $y(3)$ for the initial-value problem
$y' = \frac{\sqrt{y}}{2}, y(1) = 1$
using Heun's method (RK2) with step size $h = 1$.

24

Estimate $y(3)$ for the initial-value problem $y'' = \frac{\sqrt{y}}{2}, y(1) = 1, y'(1) = 0$ using Heun's method (RK2) with step size $h = 1$.

25

(a) Use 1 step of the RK4 method to solve for $y(x = 6)$ if $y(x = 0) = 2$ and $y' = x\sqrt{y}$.
(b) Write Python code for solving this problem and displaying the result.

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Estimate $y(t = 2)$ if $y(t = 0) = 0, dy/dt = -2y + 3\sqrt{t}$
with the RK4 method and $h = 1$.

27

(a) Use the implicit Euler method with step size $h = 1/6$ to estimate $y(1)$ given $\frac{dy}{dt} = -2y, y(0) = 3$.
(b) Find the absolute error of the estimated $y(1)$ compared to the analytic solution $y(t) = 3e^{-2t}$.
(c) Write Python code to solve for and display $y(1)$.

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Estimate $y(3)$ for the initial-value problem
$y'' = \frac{\sqrt{y}}{2}, y(1) = 5, y'(1) = 0$
using Heun's method (RK2) with step size $h = 1$.

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For the mass-spring system with driving force as given below, use 2 equal-length steps of Euler's method to estimate $x(t)$ at $t = 1$. The initial conditions at $t = 0$ are $x = 1, x' = 0$.

$$x'' + x = 3\cos(t)$$

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(a) Write a Python function, similar to the RK2 one we did in lab, to implement Euler's method for solving a system of first-order differential equations with given initial values. The first line should be

```
def rk1(f,tspan,y0,n):
```

(b) Write a Python script that uses this `rk1` and a step size h =0.01 to display $y(2)$ if $y'' = \sin(1/y)$ and $y(1) = 10, y'(1) = 3$.

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Write a Python script that solves numerically the initial-value problem
$y'' + y' + y^2 = 1, y(0) = 2, y'(0) = -2$
and plot $y(t)$ for $0 \leq t \leq 10$.

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Write a Python script that (a) numerically estimates $\Theta(2)$ if $\Theta'' = -9\Theta$ and $\Theta(0) = 1, \Theta'(0) = 0$ and (b) finds the absolute error of this estimate relative to the analytic solution $\Theta(t) = \cos(3t)$.

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(a) Set up a system of linear equations (in matrix form) using finite difference with $n = 4$ to solve the equation $y'' + y' = x + 2$ subject to the boundary conditions $y(x = 0) = 0$ and $y(x = 3) = 1$.
(b) Solve this system using Gaussian elimination or LU decomposition, and sketch $y(x)$.

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Set up a linear system in matrix form to solve the boundary-value problem $y'' + 5y' = 10$, $y(0) = 1$, $y(4) = 3$ for values of the function y(x), using second-order-accurate centered finite-difference approximations for the derivatives, with n=5. You do not need to solve the linear system.

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Set up a system of algebraic equations (in matrix form) using finite difference with $n = 5$ for $y'' - y' + y = 2$ subject to the boundary conditions $y(0) = 0$ and $y'(3) = 2$. Use second-order-accurate (centered where possible) finite difference approximations. You only need to write down the system, not to solve it.

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Set up a system of algebraic equations (in matrix form) using finite difference with $n = 5$ to solve $y'' + y' - y = 4$ subject to the boundary conditions $y(x = 0) = 0$ and $y'(x = 3) = 2$. Use second-order-accurate (centered where possible) finite difference approximations. For full credit, solve this system and sketch $y(x)$.

37

Set up a system of algebraic equations (in matrix form) using finite difference with $n = 5$ to solve $y'' + y' + y = 2$ subject to the boundary conditions $y(x = 0) = 0$ and $y'(x = 1) = 2$. For full credit, solve this system and sketch $y(x)$.

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(a) Set up a system of linear algebraic equations (in matrix form) using finite difference with $n = 5$ to solve $y'' + y = x + 4$ subject to the boundary conditions $y(x = 0) = 0$ and $y(x = 5) = 2$. Explain what the unknowns in this system represent.
(b) Write Python code to solve the same problem numerically and to plot $y(x)$.

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(a) Set up a system of linear equations (in matrix form) using finite difference with $n = 4$ to solve the equation $y'' - y = x + 3$ subject to the boundary conditions $y(x = 0) = 0$ and $y(x = 2) = 0$.
(b) Solve this system (for example, using Gaussian elimination) and sketch $y(x)$.

(c) Write Python code to solve the boundary-value problem numerically and plot $y(x)$.

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(a) Set up a system of linear equations (in matrix form) using finite difference with $n = 6$ to solve the equation $y'' + 3y' = y + 4$ subject to the boundary conditions $y(x = 0) = 0$ and $y(x = 1) = 0$.
(b) Solve this system (for example, using Gaussian elimination) and sketch the solution $y(x)$.