Practice exam 1 problems with solutions

1

Write the Taylor series of $f(x) = x^{-2}$ about $a = 1$ as the first 3 terms plus a remainder term.

Solution: The general form is $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{6}f'''(\zeta)$ for some ζ between a and x.

So here we can write $f(x) = 1 - 2(x-1) + 3(x-1)^2 - 4\zeta^{-5}(x-1)^3$ for some ζ between 1 and x.

2

Use Taylor's theorem to find a cubic polynomial $p(x)$ that approximates accurately $f(x) = x^{5/3}$ when x is close to 1.

Solution: If $f(x) = x^{5/3}$, $p(x)$ can be obtained from the first four terms of the Taylor series of $f(x)$ about $a = 1$:

$$p(x) = 1 + \frac{5}{3}(x-1) + \frac{5}{9}(x-1)^2 - \frac{5}{81}(x-1)^3.$$

$p(x)$ can also be expanded as $-\frac{5}{81}x^3 + \frac{20}{27}x^2 + \frac{10}{27}x - \frac{4}{81}$.

3

(a) Find the term of order 3 ($i = 3$) in the Taylor series of $\ln(x)$ about $a = 4$.
(b) Write Python code to sum the first 5 terms of this Taylor series for $x = 4.2$

Solution: (a) The order-i derivative ($i > 0$) of $\log(x)$ (natural logarithm) is $(-1)^{i-1}(i-1)!x^{-i}$.
The Taylor series of $\log(x)$ about a is therefore

$$\log(a) + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{ia^i}(x-a)^i.$$

Thus, if $a = 4$, the $i = 3$ term is $(x-4)^3/192$.

(b)
```
from math import log
a = 4
x = 4.2
n = 5
series_sum = log(a)
for i in range(1,n):
        series_sum = series_sum + (x - a)**i * (-1)**(i-1) / (i * a**i)
```

4

Write the Taylor series of $f(x) = \cos(x)$ about $a = \pi$ as the first 3 terms plus a remainder term.

Solution: The general form is $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{6}f'''(\zeta)$ for some ζ between a and x.

So here we can write $f(x) = -1 + \frac{1}{2}(x-\pi)^2 + \frac{1}{6}\sin(\zeta)(x-\pi)^3$ for some ζ between π and x.

5

It's given that $f(a) = b, f'(a) = c, f''(a) = d, f'''(a) = e$; all higher order derivatives of $f(x)$ are zero at $x = a$; and the function f and all its derivatives exist and are continuous between $x = a$ and $x = k$.
(a) If $a = 2, b = 1, c = -2, d = 4, e = 3, k = 5$, what is $h \equiv f(k)$?
(b) Write a Python function to solve for $f(k)$ in general. The first line should be
`def ffind (a, b, c, d, e, k):`

Solution: Based on Taylor's theorem, the general formula would be $h = b + cn + dn^2/2 + en^3/6$, where $n \equiv k - a$.
(a) Substituting the given values, $h = 1 + (-2) \cdot 3 + 4 \cdot 3^2/2 + 3 \cdot 3^3/6 = \mathbf{26.5}$
(b)
```
def ffind (a, b, c, d, e, k):
        n = k - a
        h = b + c*n + d*n**2/2 + e*n**3/6
        return h
```

6

(a) Use 5 iterations of Newton's method to find a root of the function $f(x) = x^5 - 2x + 1$ starting with $x_0 = 0$. (b) Estimate the fractional error of your result.

Solution: (a) If $f(x) = x^5 - 2x + 1$, $f'(x) = 5x^4 - 2$, and successive iterations give

0
0.5
$0.\overline{518}$
0.518790000907635
0.518790063675881
0.518790063675884
(b) Using the difference between the last two iterations to estimate the error, we get $\frac{|x_i - x_{i-1}|}{x_i} = 6 \times 10^{-15}$. Note though that even if the last two values were floating-point numbers that were identical at the precision of our computer or calculator, we should not assume that the actual fractional error is smaller than machine epsilon, or (assuming double precision) $\sim 10^{-16}$.

7

For $R = 27$, apply two iterations of Newton's method with a starting value $x_0 = 5$ to the equation $x^2 - R = 0$ to estimate $\sqrt{R}$.

Solution: $x_{i+1} \leftarrow x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - R}{2x_i} = \frac{x_i}{2} + \frac{R}{2x_i} = \frac{x_i + R/x_i}{2}$
So for $R = 27$, $x_0 = 5, x_1 = 5.2, x_2 = \mathbf{5.1962}$

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For $R = 30$, apply two iterations of Newton's method with a starting value $x_0 = 3$ to the equation $x^3 - R = 0$ to estimate $\sqrt[3]{R}$.

Solution: $x_{i+1} \leftarrow x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - R}{3x_i^2} = \frac{2x_i}{3} + \frac{R}{3x_i^2} = \frac{2x_i + R/x_i^2}{3}$
So $x_0 = 3, x_1 = 3\frac{1}{9}, x_2 = \mathbf{3.1072}$

9

Write a Python script that plots $\log_{10}(x)$ for x between 1 and 100 and labels the x and y axes.

Solution:

```
from numpy import log10, linspace
from matplotlib.pyplot import plot, xlabel, ylabel
x = linspace(1, 100, 1000) #generate enough points for a smooth curve
y = log10(x)
plot(x, y)
xlabel('x'), ylabel('y')
```

10

Suppose a is some real number that rounds to 100006 and b is another real number that rounds to 99993. Estimate the fractional error in computing $a - b$ if the subtraction is done on a computer with machine epsilon of $\epsilon = 10^{-8}$.

Solution: The absolute roundoff error for $a - b$ can be as high as around $\epsilon(|a| + |b|)$. The fractional error is therefore bounded by $\epsilon\frac{|a|+|b|}{|a-b|}$, where the rounded values can be used to estimate $|a|, |b|, |a-b|$. This gives $10^{-8}\frac{|100006|+|99993|}{|100006-99993|}$, or around 1.5×10^{-4}.

11

To 3 significant digits, find the absolute and relative errors of approximating 100 lbf by 442 N (conversion factor: 4.4482216152605 N per lbf). Give correct units.

Solution: Absolute error: $|100 - 442/4.4482216152605| =$ **0.634 lbf** (or 2.82 N)

Relative error: $0.634/|100| = \mathbf{6.34 \times 10^{-3}}$ (dimensionless)

12

To 3 significant digits, find the absolute and relative errors of approximating 1000 lbm by 453 kg (conversion factor: 0.45359237 kg per lb). Make sure to give correct units.

Solution: Absolute error: $|1000 - 453/0.45359237| =$ **1.31 lb** (or 0.592 kg)

Relative error: $1.31/|1000| = \mathbf{1.31 \times 10^{-3}}$ (dimensionless)

13

To 3 significant digits, find the absolute and relative errors of approximating $\cos(0.4)$ by the sum of the first 3 nonzero terms of the Maclaurin series of cosine.

Solution: The Taylor series approximation is $1 - (0.4)^2/2 + (0.4)^4/24$. Compared to a more accurate value of the cosine, it has absolute error 5.67×10^{-6} and relative error 6.16×10^{-6}.

14

To 5 significant digits, find the absolute and relative errors of approximating the number $\gamma = 0.5772156649\ldots$ by 0.577

Solution: Absolute error: 2.1566×10^{-4}; relative error: 3.7363×10^{-4}. (This is the Euler-Mascheroni constant from math.)

15

To 4 significant digits, find the absolute and relative errors of approximating the number $\alpha = 0.0072973525664\ldots$ by $\frac{1}{137}$.

Solution: Absolute error: 1.918×10^{-6}; relative error: 2.628×10^{-4}. (This is the fine-structure constant from physics.)

16

(a) What is decimal 8.5 in base 2?
(b) What is binary 11100 in base 10?

Solution: (a) 8.5 is 2^3 plus 1/2, or binary 1000.1
(b) Binary 11100 is $4 + 8 + 16$, or 28.

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(a) What is decimal 0.625 in base 2?
(b) What is binary 10111 in base 10?

Solution: (a) 0.625 is 1/2 plus 1/8, or binary 0.101
(b) Binary 10111 is $1 + 2 + 4 + 16$, or 23.

18

If $\mathbf{A} = \mathbf{LU}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 & 1 & 4 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 44 \\ 154 \\ 256 \\ 74 \end{pmatrix}$, find $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$.

Solution: Given the LU factorization of $\mathbf{A}$, we solve $\mathbf{Ly} = \mathbf{b}$ by forward substitution to get $\mathbf{y} = \begin{pmatrix} 44 \\ 22 \\ 14 \\ 16 \end{pmatrix}$, followed by solving $\mathbf{Ux} = \mathbf{y}$ by back substitution to get $\mathbf{x} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 4 \end{pmatrix}$. Note that it isn't necessary to compute $\mathbf{A}$.

19

(a) Find the LU decomposition of the matrix $\mathbf{N} = \begin{pmatrix} 2 & -2 & 0 \\ x & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$.

(b) For what value of x does $\mathbf{N}$ have no inverse? Explain.

Solution: (a) The factors are $\mathbf{U} = \begin{pmatrix} 2 & -2 & 0 \\ 0 & x+2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{x}{2} & 1 & 0 \\ 0 & -\frac{1}{x+2} & 1 \end{pmatrix}$.

(b) The matrix determinant is $6x + 12$, which is zero only when $x = -2$, indicating that the matrix has no inverse then (which can also be seen from the second row being a multiple of the first, and thus the rows are not linearly independent).

20

Given $\mathbf{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 3 & 0 \\ -2 & 1 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{Ax} = \mathbf{b}$,

(a) Find $\mathbf{L}$, $\mathbf{U}$, $\mathbf{P}$ for the LU decomposition of $\mathbf{A}$ with row pivoting.

(b) Find $\mathbf{x}$.

Solution: (a) Using Gauss elimination with row pivoting, the steps are

$$\begin{pmatrix} 3 & -2 & 1 & (1) \\ 0 & 3 & 0 & (2) \\ -2 & 1 & 0 & (3) \end{pmatrix} \to \begin{pmatrix} 3 & -2 & 1 & (1) \\ 0| & 3 & 0 & (2) \\ -\frac{2}{3}| & -\frac{1}{3} & \frac{2}{3} & (3) \end{pmatrix} \to \begin{pmatrix} 3 & -2 & 1 & (1) \\ 0| & 3 & 0 & (2) \\ -\frac{2}{3} & -\frac{1}{9}| & \frac{2}{3} & (3) \end{pmatrix}$$

No row pivoting is actually done in this case.

The factors are $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{3} & -\frac{1}{9} & 1 \end{pmatrix}$, $\mathbf{U} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Given the LU factorization of $\mathbf{A}$, we solve $\mathbf{Ly} = \mathbf{Pb}$ by forward substitution to get $\mathbf{y} = \begin{pmatrix} 2 \\ 0 \\ 4/3 \end{pmatrix}$, followed by solving $\mathbf{Ux} = \mathbf{y}$ by back substitution to get $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

21

(a) Use Gauss elimination with row pivoting to find $\mathbf{L}$ and $\mathbf{U}$ factors for the matrix $\mathbf{M} = \begin{pmatrix} 0 & 3 & 4 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(b) How is the product $\mathbf{LU}$ related to $\mathbf{M}$? Explain.

Solution: (a) Row pivoting in this case interchanges the first and second rows of $\mathbf{M}$.

The factors are $\mathbf{U} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 4 & 4 \\ 0 & 0 & -2/3 & -5/3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$, $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) The product $\mathbf{LU}$ is the original matrix $\mathbf{M}$, but with the rows interchanged in the same sequence followed in the row pivoting steps. So $\mathbf{LU} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 4 & 4 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

22

Suppose the Cholesky factor for a symmetric matrix $\mathbf{A}$ is $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) What is $\mathbf{A}$?

(b) Solve $\mathbf{Ax} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$ for $\mathbf{x}$, using forward and back substitution.

Solution: (a) By definition of the Cholesky factorization, we have $\mathbf{A} = \mathbf{LL}^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(b) Since $\mathbf{LL}^T\mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$, forward substitution on $\mathbf{Ly} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$ gives $\mathbf{y} = \begin{pmatrix} 4 \\ -1/2 \\ 3 \end{pmatrix}$, followed by back substitution on $\mathbf{L}^T\mathbf{x} = \mathbf{y}$ to get $\mathbf{x} = \begin{pmatrix} 5/4 \\ -1/4 \\ 3 \end{pmatrix}$.

23

Let $\mathbf{A}$ be a symmetric positive definite matrix whose Cholesky factor is

$$\mathbf{L} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ -1 & 2 & 1 & 2 \end{pmatrix}.$$

(a) Solve the system $\mathbf{Ax} = \mathbf{b} = \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}^T$. Show and explain all steps.
(b) Find $\mathbf{A}$.

Solution: (a) Solve the lower triangular system $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$ using forward substitution, then solve the upper triangular system $\mathbf{L}^T\mathbf{x} = \mathbf{y}$ using back substitution. Note that it isn't necessary to compute $\mathbf{A}$ for this part. (b) Matrix multiply (by definition, $\mathbf{A} = \mathbf{LL}^T$).
In this case, we get $\mathbf{y} = \begin{pmatrix} 0 & 1 & 2/3 & -4/3 \end{pmatrix}^T$, $\mathbf{x} = \begin{pmatrix} 1/6 & 17/9 & 4/9 & -2/3 \end{pmatrix}^T$, and

$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 4 & -2 \\ -2 & 2 & -1 & 3 \\ 4 & -1 & 14 & 3 \\ -2 & 3 & 3 & 10 \end{pmatrix}.$$