

## Practice exam 2 problems

### 1

In each case, determine whether  $\mathbf{v}$  is an eigenvector of  $\mathbf{A}$ . Explain briefly how you know. If it is an eigenvector, state the eigenvalue.

(a)  $\mathbf{A} = \begin{pmatrix} -7 & 5 & -2 \\ -14 & 6 & 4 \\ -11 & 5 & 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$

(b)  $\mathbf{A} = \begin{pmatrix} 11 & 3 & -12 \\ -6 & -4 & 6 \\ 9 & 3 & -10 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$

(c)  $\mathbf{A} = \begin{pmatrix} -1 & 2 & 4 \\ -8 & 1 & -4 \\ 4 & -2 & -1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$

### 2

(a) What's the largest (in absolute value) eigenvalue of  $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ ?

(b) Use 2 iterations of the power method, starting with  $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ , to estimate the associated eigenvector.

### 3

Use 2 iterations of the power method with initial guess  $\mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  to estimate

the largest eigenvalue of  $\mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}.$

## 4

- (a) What are the eigenvalues of  $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 1 & 5 \end{pmatrix}$ ?
- (b) What are the eigenvectors of  $\mathbf{A}$ ? Scale them so that the first component of each is 1.
- (c) If this  $\mathbf{A}$  is the coefficient matrix  $\mathbf{M}^{-1}\mathbf{K}$  for a system of linear oscillators, what are the periods of the system's modes of oscillation?
- (d) Suppose you're given a  $10 \times 10$  matrix  $\mathbf{B}$  that has only positive real eigenvalues. Write Python code to find and display the eigenvector associated with the smallest eigenvalue of  $\mathbf{B}$ .

## 5

Find the solution  $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$  to the initial-value problem  $\begin{matrix} x_1'(t) = 8x_1(t) + x_2(t) \\ x_2'(t) = x_1(t) + 8x_2(t) \end{matrix}$ ,  
 $x_1(0) = 3$   
 $x_2(0) = 2$  .

## 6

Find the general solution  $x_1(t), x_2(t)$  to the system of second-order differential equations

$$\begin{aligned} 0 &= x_1'' + 2x_1 + x_2 \\ 0 &= x_2'' + x_1 + 3x_2. \end{aligned}$$

## 7

For a two-story shear building with  $k_1 = 400, k_2 = 300, m_1 = 4, m_2 = 3$ , (a) Write the mass matrix  $\mathbf{M}$  and stiffness matrix  $\mathbf{K}$ , (b) Write Python code for finding all the oscillation frequencies.

## 8

Solving a shear building problem, you obtain the following result in Python for `(lambda, v) = scipy.linalg.eig(A):`

```
v =
-2.2801e-01  5.7735e-01  6.5654e-01 -4.2853e-01
-4.2853e-01  5.7735e-01 -2.2801e-01  6.5654e-01
-5.7735e-01  4.7345e-17 -5.7735e-01 -5.7735e-01
-6.5654e-01 -5.7735e-01  4.2853e-01  2.2801e-01
lambda =
1.2061, 10.000, 23.473, 35.320
```

- (a) What is the period of the fundamental mode of oscillation?  
 (b) What is special about the eigenvector associated with this mode?

## 9

Approximate the first two derivatives of  $y(x)$  at the given points using second order accurate finite differences (centered if possible).

|       |    |   |   |   |
|-------|----|---|---|---|
| $x$   | -1 | 0 | 1 | 2 |
| $y$   | 2  | 2 | 3 | 4 |
| $y'$  | -  | - | - | - |
| $y''$ | -  | - | - | - |

## 10

Given the following data, approximate the first four derivatives of  $f(x)$  at  $x = 1$  using centered finite differences.

|        |    |   |   |   |   |
|--------|----|---|---|---|---|
| $x$    | -1 | 0 | 1 | 2 | 3 |
| $f(x)$ | 1  | 2 | 4 | 3 | 2 |

## 11

- (a) Find  $D_0^2$  for Richardson extrapolation to estimate the derivative of  $f(x)$  at  $x = 1.1$ , with initial step size  $h_0 = 0.4$ . Use the function values given below, and show at least 7 significant figures.  
 (b) What is the estimated fractional error in this obtained  $f'(x)$ ?

|        |       |       |       |       |       |        |        |        |        |        |        |
|--------|-------|-------|-------|-------|-------|--------|--------|--------|--------|--------|--------|
| $x$    | 0.5   | 0.6   | 0.7   | 0.8   | 0.9   | 1      | 1.1    | 1.2    | 1.3    | 1.4    | 1.5    |
| $f(x)$ | 61255 | 67322 | 74199 | 81907 | 90488 | 100000 | 110513 | 122106 | 134870 | 148902 | 164311 |

## 12

- (a) Find  $D_0^2$  for Richardson extrapolation to estimate the derivative of  $f(x)$  at  $x = 1$ , with initial step size  $\Delta_0 = 0.4$ . Use the function values given below.  
 (b) What is the estimated relative error in this estimated  $f'(x)$ ?

|        |       |       |       |       |       |        |        |        |        |        |        |
|--------|-------|-------|-------|-------|-------|--------|--------|--------|--------|--------|--------|
| $x$    | 0.5   | 0.6   | 0.7   | 0.8   | 0.9   | 1      | 1.1    | 1.2    | 1.3    | 1.4    | 1.5    |
| $f(x)$ | 70711 | 73602 | 77906 | 83651 | 90953 | 100000 | 111053 | 124456 | 140646 | 160169 | 183712 |

## 13

Consider the composite trapezoid rule approximation  $T_{78}$  of  $\int_0^1 x^{95} dx$ .

- (a) Does  $T_{78}$  overestimate or underestimate the exact value? Explain how you know.

(b) Find a bound for the absolute error in  $T_{78}$  using the result that  $\text{Error}(T_N) \leq \frac{M(b-a)^3}{12N^2}$ , where  $M$  is the least upper bound for all absolute values of the second derivatives over  $[a, b]$  of the function being integrated.

## 14

- (a) Estimate the integral  $\int_0^2 x^{10} dx$  using the simple Simpson's rule.  
 (b) Find the minimum number  $N$  of equal-width subdivisions needed to guarantee that the absolute error in the composite Simpson rule estimate for this integral is under  $10^{-7}$  using the error bound  $\frac{K_4(b-a)^5}{2880N^4}$ .

## 15

- Estimate  $I = \int_2^4 y(x) dx$  given the values below (a) using the simple Simpson rule,  
 (b) using the composite Simpson rule with  $n = 2$  equal-length intervals.  
 (c) Based on (a) and (b), estimate what  $n$  would be needed for the absolute error in estimating  $I$  using the composite Simpson rule to be under  $10^{-3}$ .

|        |     |     |    |     |    |
|--------|-----|-----|----|-----|----|
| $x$    | 2   | 2.5 | 3  | 3.5 | 4  |
| $y(x)$ | 100 | 67  | 50 | 40  | 33 |

## 16

- Estimate the integral of  $e^{(x^2)}$  between  $a = 0$  and  $b = 1$  using Simpson's rule with  
 (a) one subinterval,  
 (b) two subintervals.  
 (c) Estimate your fractional error based on the difference between the two results.

## 17

An irregular lot is 100 ft long, and its width as measured at various points is as follows:

|               |    |    |    |    |     |
|---------------|----|----|----|----|-----|
| position (ft) | 0  | 25 | 50 | 75 | 100 |
| width (ft)    | 10 | 15 | 20 | 23 | 25  |

Estimate the lot's area as accurately as possible using the composite Simpson rule.

## 18

Starting from standstill at  $t = 0$ , the acceleration of an elevator is measured as

|                   |   |     |     |     |     |      |      |
|-------------------|---|-----|-----|-----|-----|------|------|
| $t(\text{s})$     | 0 | 5   | 10  | 15  | 20  | 23   | 30   |
| $a(\text{m/s}^2)$ | 0 | 2.2 | 3.6 | 5.2 | 6.0 | -4.0 | -8.1 |

- (a) Estimate the velocity at  $t = 20$  using the composite trapezoid rule.
- (b) Write a Python script to estimate the velocity at  $t = 20$  by fitting an interpolating polynomial to the data and integrating it.

## 19

Write Python code to estimate the probability of falling between  $x = 1$  and  $x = 2$  for the probability density function  $p(x) = e^{-(e^{-x}+x)}$ .

## 20

Calculate the Romberg integration estimator  $R_0^2$  of

$$\int_0^1 \frac{x^3}{x+1} dx.$$

## 21

- (a) Use Romberg integration with  $j_{\max} = 2$  to estimate

$$I = \int_0^2 \frac{x}{x+3} dx.$$

- (b) Estimate the relative error based on the difference between your two most accurate estimates.

## 22

- (a) Fill in the first 3 levels of Romberg integration to estimate the integral of  $\sqrt{x}$  for  $x$  between 0 and 3.
- (b) Estimate the absolute error based on the difference between the two most accurate estimates above.
- (c) Find the actual absolute error of the most accurate estimate above compared to the analytic integral.

## 23

- (a) Use Romberg integration with  $j_{\max} = 3$  to estimate  $I = \int_0^1 (e^x - 4x)dx$ .
- (b) Find the solution analytically, and based on this calculate the fractional error of part a.

## 24

Use Romberg integration with  $j_{\max} = 2$  to estimate  $I = \int_0^2 \frac{x^2}{x^2+3}dx$ .