

Practice exam 2 problems with solutions

1

In each case, determine whether $\mathbf{v}$ is an eigenvector of $\mathbf{A}$. Explain briefly how you know. If it is an eigenvector, state the eigenvalue.

$$\begin{aligned} \text{(a) } \mathbf{A} &= \begin{pmatrix} -7 & 5 & -2 \\ -14 & 6 & 4 \\ -11 & 5 & 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}. \\ \text{(b) } \mathbf{A} &= \begin{pmatrix} 11 & 3 & -12 \\ -6 & -4 & 6 \\ 9 & 3 & -10 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}. \\ \text{(c) } \mathbf{A} &= \begin{pmatrix} -1 & 2 & 4 \\ -8 & 1 & -4 \\ 4 & -2 & -1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}. \end{aligned}$$

Solution: In each case, you need to compute the matrix product $\mathbf{A}\mathbf{v}$. If it is a multiple of $\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ and the multiplier is the eigenvalue.

Here, in (a) and (b) $\mathbf{v}$ isn't an eigenvector, whereas in (c), $\mathbf{v}$ is an eigenvector with an eigenvalue of -1 .

2

- (a) What's the largest (in absolute value) eigenvalue of $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 4 \end{pmatrix}$?
- (b) Use 2 iterations of the power method, starting with $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, to estimate the associated eigenvector.

Solution: (a) Since $\mathbf{B}$ is triangular, the elements on its main diagonal are its eigenvalues. So the largest eigenvalue is **5**.

(b) Dividing at each step by the largest absolute values in $\mathbf{v}$ (the infinity norm), we have $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2/3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 8/17 \end{pmatrix}$.

3

Use 2 iterations of the power method with initial guess $\mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to estimate

the largest eigenvalue of $\mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}$.

Solution: $\lambda_1 = \|\mathbf{A}\mathbf{v}_0\|_2 = \sqrt{17} = 4.123$
 $\mathbf{v}_1 = \mathbf{A}\mathbf{v}_0/\lambda_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} / \sqrt{17}$
 $\lambda_2 = \|\mathbf{A}\mathbf{v}_1\|_2 = \sqrt{\frac{293}{17}} = \mathbf{4.152}.$

4

- What are the eigenvalues of $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 1 & 5 \end{pmatrix}$?
- What are the eigenvectors of $\mathbf{A}$? Scale them so that the first component of each is 1.
- If this $\mathbf{A}$ is the coefficient matrix $\mathbf{M}^{-1}\mathbf{K}$ for a system of linear oscillators, what are the periods of the system's modes of oscillation?
- Suppose you're given a 10×10 matrix $\mathbf{B}$ that has only positive real eigenvalues. Write Python code to find and display the eigenvector associated with the smallest eigenvalue of $\mathbf{B}$.

Solution: (a) For $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 1 & 5 \end{pmatrix}$, the characteristic polynomial is $(6 - \lambda)(5 - \lambda) - 1$, with roots at $\frac{11 \pm \sqrt{121 - 116}}{2}$, or $\frac{11 \pm \sqrt{5}}{2}$.
 (b) They are $(1, \frac{-1 \pm \sqrt{5}}{2})$.
 (c) 2π divided by the eigenvalue square roots, or 2.4424 and 3.0015
 (d)

```
from scipy.linalg import eig
from numpy import argmin (d, v) = eig(B)
v[:, argmin(d)]
```

5

Find the solution $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ to the initial-value problem $\begin{matrix} x_1'(t) = 8x_1(t) + x_2(t) \\ x_2'(t) = x_1(t) + 8x_2(t) \end{matrix}$,
 $\begin{matrix} x_1(0) = 3 \\ x_2(0) = 2 \end{matrix}$.

Solution: We can write the system as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with coefficient matrix $\mathbf{A} = \begin{pmatrix} 8 & 1 \\ 1 & 8 \end{pmatrix}$, which has eigenvalues $\lambda = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$ and eigenvectors $\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. The general solution is therefore $\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{9t}$. The initial conditions at $t = 0$ require $c_1 = -\frac{1}{2}, c_2 = \frac{5}{2}$.

6

Find the general solution $x_1(t), x_2(t)$ to the system of second-order differential equations

$$\begin{aligned} 0 &= x_1'' + 2x_1 + x_2 \\ 0 &= x_2'' + x_1 + 3x_2. \end{aligned}$$

Solution: This can be written in matrix form as $\mathbf{x}'' = -\mathbf{A}\mathbf{x}$, with $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. As we saw in class, the general solution is $\mathbf{x}(t) = c_1 \sin(\sqrt{\lambda_1}t)\mathbf{v}_1 + d_1 \cos(\sqrt{\lambda_1}t)\mathbf{v}_1 + c_2 \sin(\sqrt{\lambda_2}t)\mathbf{v}_2 + d_2 \cos(\sqrt{\lambda_2}t)\mathbf{v}_2$, where $\lambda_i, \mathbf{v}_i$ are the eigenvalues and corresponding eigenvectors of $\mathbf{A}$ and the c_i and d_i can be any numbers. For this case the eigenvalues are $2.5 \pm \sqrt{1.25}$ and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0.5 \pm \sqrt{1.25} \end{pmatrix}$ (or any nonzero multiples of those vectors).

7

For a two-story shear building with $k_1 = 400, k_2 = 300, m_1 = 4, m_2 = 3$, (a) Write the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$, (b) Write Python code for finding all the oscillation frequencies.

Solution: (a) $\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

$$\mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} = \begin{pmatrix} 700 & -300 \\ -300 & 300 \end{pmatrix}$$

(b)

```
from numpy import array, diag, sqrt, pi
from numpy.linalg import solve
from scipy.linalg import eig
m1 = 4; m2 = 3;
k1 = 400; k2 = 300;
M = diag([m1, m2]); K = array([[k1+k2, -k2], [-k2, k2]])
A = solve(M, K)
(1, v) = eig(A) # vector with all eigenvalues
f = sqrt(1) ./ (2*pi) #oscillation frequencies (cycles per unit time;
periods are 1 / f)
```

8

Solving a shear building problem, you obtain the following result in Python for `(lambda, v) = scipy.linalg.eig(A):`

```
v =
-2.2801e-01  5.7735e-01  6.5654e-01 -4.2853e-01
-4.2853e-01  5.7735e-01 -2.2801e-01  6.5654e-01
-5.7735e-01  4.7345e-17 -5.7735e-01 -5.7735e-01
-6.5654e-01 -5.7735e-01  4.2853e-01  2.2801e-01
lambda =
1.2061, 10.000, 23.473, 35.320
```

- (a) What is the period of the fundamental mode of oscillation?
- (b) What is special about the eigenvector associated with this mode?

Solution: (a) 2π divided by the square root of the smallest eigenvalue, or **5.721** time units

(b) Since the smallest eigenvalue is the first one, the associated eigenvector is given by the first column of **v**. It is the only eigenvector which has all its elements the same sign. This corresponds to an oscillation where the whole building sways to each side together, with the higher stories swaying the farthest.

9

Approximate the first two derivatives of $y(x)$ at the given points using second order accurate finite differences (centered if possible).

x	-1	0	1	2
y	2	2	3	4
y'	-	-	-	-
y''	-	-	-	-

Solution: Using the formulas in the lecture notes and homework, the values are

x	-1	0	1	2
y	2	2	3	4
y'	-0.5	0.5	1	1
y''	2	1	0	-1

10

Given the following data, approximate the first four derivatives of $f(x)$ at $x = 1$ using centered finite differences.

x	-1	0	1	2	3
$f(x)$	1	2	4	3	2

Solution: Using the formulas given in the lecture notes (with $\Delta x = 1$):

$$f'(1) \approx (f(2) - f(0))/2 = 0.5$$

$$f''(1) \approx (f(2) - 2 \cdot f(1) + f(0))/1^2 = -3$$

$$f'''(1) \approx (-f(-1) + 2 \cdot f(0) - 2 \cdot f(2) + f(3))/(2 \cdot 1^3) = -0.5$$

$$f''''(1) \approx (f(-1) - 4 \cdot f(0) + 6 \cdot f(1) - 4 \cdot f(2) + f(3))/1^4 = 7$$

11

(a) Find D_0^2 for Richardson extrapolation to estimate the derivative of $f(x)$ at $x = 1.1$, with initial step size $h_0 = 0.4$. Use the function values given below, and show at least 7 significant figures.

(b) What is the estimated fractional error in this obtained $f'(x)$?

x	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$f(x)$	61255	67322	74199	81907	90488	100000	110513	122106	134870	148902	164311

Solution:

$$\begin{aligned} D_0^1 &= (f(1.4) - f(0.6))/0.8 = 112640 \\ D_1^0 &= (f(1.2) - f(0.8))/0.4 = 110393.3 \\ D_2^0 &= (f(1.1) - f(0.9))/0.2 = 110388.0 \\ D_0^1 &= (4D_1^0 - D_0^0)/3 = 110955 \\ D_1^1 &= (4D_2^0 - D_1^0)/3 = 110388.3 \\ D_2^1 &= (16D_1^1 - D_0^1)/15 = 110530 \end{aligned}$$

The fractional error can be estimated as $\frac{|D_1^1 - D_0^2|}{|D_0^2|} = \mathbf{3.0 \times 10^{-6}}$.

12

(a) Find D_0^2 for Richardson extrapolation to estimate the derivative of $f(x)$ at $x = 1$, with initial step size $\Delta_0 = 0.4$. Use the function values given below.

(b) What is the estimated relative error in this estimated $f'(x)$?

x	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$f(x)$	70711	73602	77906	83651	90953	100000	111053	124456	140646	160169	183712

Solution: We have

$$D_0^0 = (f(1.4) - f(0.6))/0.8 = 108209$$

$$D_1^0 = (f(1.2) - f(0.8))/0.4 = 102012$$

$$D_2^0 = (f(1.1) - f(0.9))/0.2 = 100500$$

$$D_0^1 = (4D_1^0 - D_0^0)/3 = 99947$$

$$D_1^1 = (4D_2^0 - D_1^0)/3 = 99996$$

$$D_2^1 = (16D_1^1 - D_0^1)/15 = \mathbf{99999}$$

The relative error can be estimated as $|\frac{D_2^1 - D_0^2}{D_0^2}| = \mathbf{3 \times 10^{-5}}$.

13

Consider the composite trapezoid rule approximation T_{78} of $\int_0^1 x^{95} dx$.

(a) Does T_{78} overestimate or underestimate the exact value? Explain how you know.

(b) Find a bound for the absolute error in T_{78} using the result that $\text{Error}(T_N) \leq \frac{M(b-a)^3}{12N^2}$, where M is the least upper bound for all absolute values of the second derivatives over $[a, b]$ of the function being integrated.

Solution: The second derivative is $95 \cdot 94 \cdot x^{93}$, with a maximum absolute value in the interval of $95 \cdot 94$. So the error bound is

$$\frac{95 \cdot 94 \cdot (1-0)^3}{12 \cdot 78^2} = \mathbf{0.12}$$

(the actual error is considerably less, however).

Because the second derivative is always positive, the composite trapezoid rule will in this case **overestimate** the integral.

14

(a) Estimate the integral $\int_0^2 x^{10} dx$ using the simple Simpson's rule.

(b) Find the minimum number N of equal-width subdivisions needed to guarantee that the absolute error in the composite Simpson rule estimate for this integral is under 10^{-7} using the error bound $\frac{K_4(b-a)^5}{2880N^4}$.

Solution: (a) $(2-0)(f(0) + 4f(1) + f(2))/6 = \mathbf{342\frac{2}{3}}$.

(b) The 4th derivative for this function is $5040x^6$, which gets as large as 322560 when $x = 2$. The error bound gives for N a minimum of $\left(\frac{322560(2-0)^5}{(2880)(10^{-7})}\right)^{1/4} = 435.10$, so we need N of at least **436**.

15

Estimate $I = \int_2^4 y(x) dx$ given the values below (a) using the simple Simpson rule,

(b) using the composite Simpson rule with $n = 2$ equal-length intervals.

(c) Based on (a) and (b), estimate what n would be needed for the absolute error in estimating I using the composite Simpson rule to be under 10^{-3} .

x	2	2.5	3	3.5	4
$y(x)$	100	67	50	40	33

Solution: (a) $S_1 = (4-2) \times (y(2) + 4y(3) + y(4))/6 = \mathbf{111}$

(b) $S_2 = (4-2) \times (y(2) + 4y(2.5) + 2y(3) + 4y(3.5) + y(4))/12 = \mathbf{110\frac{1}{6}}$

(c) The difference between S_1 and S_2 , which we can use as a rough estimate of

the error of S_1 , is $\frac{5}{6}$, or 833×10^{-3} . To get the absolute error down by a factor of 833, to 10^{-3} , we expect to have to increase n by a factor of $833^{1/4} = 5.4$, since the Simpson rule error is typically proportional to n^{-4} . So $n = 6$ might be enough.

16

Estimate the integral of $e^{(x^2)}$ between $a = 0$ and $b = 1$ using Simpson's rule with

- (a) one subinterval,
- (b) two subintervals.
- (c) Estimate your fractional error based on the difference between the two results.

Solution: The integrand function values needed are

x	0	0.25	0.5	0.75	1
$f(x)$	1	1.0645	1.2840	1.7551	2.7183

- (a) Using the values at 0, 0.5, 1: $S_1 = 1.4757$
- (b) Using all the above values: $S_2 = 1.4637$
- (c) The fractional difference between S_1 and S_2 , $\frac{|S_1 - S_2|}{|S_2|}$ (S_2 being the more accurate estimate), which we can use as an estimate of the error in S_1 , is 8.2×10^{-3} . Given that the error in the composite Simpson's rule scales as n^{-4} for smooth functions and large enough n , we could extrapolate to estimate that the fractional error in S_2 might be only 1/16 of that, or 5×10^{-4} , which in this case actually turns out to be only a moderate underestimate (the actual fractional error in S_2 is 7.2×10^{-4}).

17

An irregular lot is 100 ft long, and its width as measured at various points is as follows:

position (ft)	0	25	50	75	100
width (ft)	10	15	20	23	25

Estimate the lot's area as accurately as possible using the composite Simpson rule.

Solution: With the given data, we can apply Simpson's rule to the two halves of the lot (0 to 50 ft and 50 to 100 ft). The result is $S_2 = \mathbf{1892 \text{ ft}^2}$.

18

Starting from standstill at $t = 0$, the acceleration of an elevator is measured as

$t(\text{s})$	0	5	10	15	20	23	30
$a(\text{m/s}^2)$	0	2.2	3.6	5.2	6.0	-4.0	-8.1

- (a) Estimate the velocity at $t = 20$ using the composite trapezoid rule.
- (b) Write a Python script to estimate the velocity at $t = 20$ by fitting an interpolating polynomial to the data and integrating it.

Solution: (a) The velocity is equal to the integral of the acceleration from $t = 0$ to $t = 20$. Using the composite trapezoid rule, this can be estimated as $(5/2) * (0 + 2 * 2.2 + 2 * 3.6 + 2 * 5.2 + 6.0) = 70$ m/s.

(b)

```
from numpy import concatenate, linspace, array
from numpy import polyfit, polyval, polyint
t = concatenate((linspace(0, 20, 5), [23, 30]))
a = array([0, 2.2, 3.6, 5.2, 6.0, -4.0, -8.1])
p = polyfit(t, a, len(t)-1)
ip = polyint(p)
v = polyval(ip, 20)
(The result is 66.8 m/s.)
```

19

Write Python code to estimate the probability of falling between $x = 1$ and $x = 2$ for the probability density function $p(x) = e^{-(e^{-x}+x)}$.

Solution: The solution is equal to the definite integral $\int_1^2 p(x) dx$, which can be found numerically in Python:

```
from numpy import exp
from scipy.integrate import quad
p = lambda x: exp(-(exp(-x) + x))
I = quad (p, 1, 2)
I = I[0] (The result is 0.181)
```

20

Calculate the Romberg integration estimator R_0^2 of

$$\int_0^1 \frac{x^3}{x+1} dx.$$

Solution: For $f(x) = \frac{x^3}{x+1}$, $a = 0$, $b = 1$, we have

$$\begin{aligned}
R_0^0 &= (f(0) + f(1))/2 = \frac{1}{4} \\
R_1^0 &= (f(0) + 2 * f(0.5) + f(1))/4 = \frac{1}{6} \\
R_2^0 &= (f(0) + 2 * f(0.25) + 2 * f(0.5) + 2 * f(0.75) + f(1))/8 = 0.14673 \\
R_0^1 &= (4R_1^0 - R_0^0)/3 = 901/5400 = 0.13\bar{8} \\
R_1^1 &= (4R_2^0 - R_1^0)/3 = 0.14008 \\
R_0^2 &= (16R_1^1 - R_0^1)/15 = \mathbf{0.14016} \\
\text{In this case the true value is } \frac{5}{6} - \log(2) &\approx 0.14019.
\end{aligned}$$

21

(a) Use Romberg integration with $j_{\max} = 2$ to estimate

$$I = \int_0^2 \frac{x}{x+3} dx.$$

(b) Estimate the relative error based on the difference between your two most accurate estimates.

Solution:

0.4 0.46667 **0.46751**

0.45 0.46746

0.46310

Estimated relative error: $\frac{|R_1^1 - R_0^2|}{|R_0^2|} = \frac{|0.46746 - 0.46751|}{|0.46751|} = \mathbf{1.1 \times 10^{-4}}$.

22

(a) Fill in the first 3 levels of Romberg integration to estimate the integral of $\sqrt{x}$ for x between 0 and 3.

(b) Estimate the absolute error based on the difference between the two most accurate estimates above.

(c) Find the actual absolute error of the most accurate estimate above compared to the analytic integral.

Solution: (a) The values are

2.598076 3.315515 **3.417804**

3.136155 3.411411

3.342597

(b) $|3.411411 - 3.417804| = 0.00639$

(c) $|3.417804 - 2\sqrt{3}| = 0.0463$ (in this case the error is underestimated at this stage)

23

(a) Use Romberg integration with $j_{\max} = 3$ to estimate $I = \int_0^1 (e^x - 4x) dx$.

(b) Find the solution analytically, and based on this calculate the fractional

error of part a.

Solution: (a) For estimating $I = \int_0^1 (e^x - 4x) dx$, the needed function values for the trapezoid rule estimates up to $j_{\max} = 3$ are those at $2^{j_{\max}} + 1 = 9$ equally spaced points:

x	0	1/8	1/4	3/8	1/2	5/8	3/4	7/8	1
f	1	0.633148	0.284025	-0.045009	-0.351279	-0.631754	-0.883000	-1.101125	-1.281718

Based on these, the trapezoid rule estimates are

N	1	2	4	8
T	-0.14086	-0.24607	-0.27278	-0.27948

Using these for the $j = 0$ column in the table of Romberg integral estimates, we get

$i \backslash j$	0	1	2	3
0	-0.14086	-0.28114	-0.28172	-0.28172
1	-0.24607	-0.28168	-0.28172	
2	-0.27278	-0.28172		
3	-0.27948			

The best estimate from this is T_0^3 or **-0.28172**.

(b) The analytic integral is $e^x - 2x^2 \Big|_0^1 = e - 3$, which is only about 3×10^{-10} lower than the Romberg value.

24

Use Romberg integration with $j_{\max} = 2$ to estimate $I = \int_0^2 \frac{x^2}{x^2+3} dx$.

Solution:

0.57143 0.52381 **0.51502**

0.53571 0.51557

0.52060