11. Solving equations

Outline

- Single variable
  - Definition
  - Methods: Newton's, bisection, secant, false position
- Multi-variable (just Newton's method)

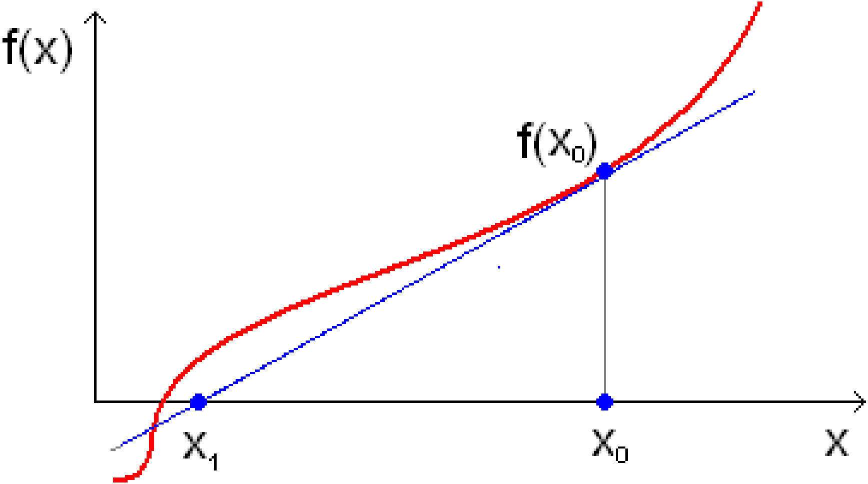
Definition

- In general form, an equation in one unknown, x, can be written as $g(x) = h(x)$, where g and h are some functions.
- The standard form is $f(x) = 0$, which can be obtained from the general form by setting $f = g - h$
- A solution to the standard-form problem is denoted a *root* of f
- If f is linear in x there will only be one root (or none, or infinitely many), but for nonlinear f there can be multiple roots, e.g. $x^2 - 1 = 0$

Newton's method

- Start with an initial guess x_0 for the root
- Iteration is $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
- Can be derived from Taylor series
  - 'Close enough' to a root, error after n iterations is proportional to $C^{(2^n)}$, where C depends on the problem and starting point (quadratic convergence)
  - In practice, can estimate error (and choose to stop iterating once the error is small snough) from the difference between successive x_i, or from the magnitude of $f(x_n)$

Illustration



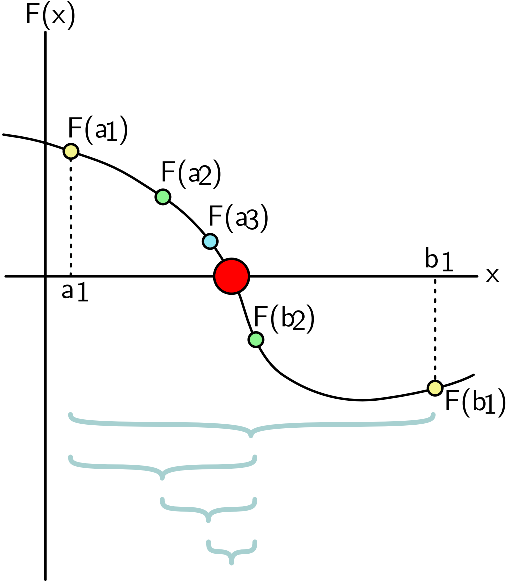
Example

$f(x) = x^3 - 3x + 1$ (has 3 roots)
Newton method iteration would be $x_{i+1} = x_i - \frac{x_i^3 - 3x_i + 1}{3x_i^2 - 3}$
Starting with $x_0 = 2$, we get
$x_1 = \frac{5}{3}, x_2 \approx 1.5486, x_3 \approx 1.5324, x_4 \approx 1.5321$
Note that if we start from another point, e.g. $x = 0$, we would converge to a different root, and from some points, e.g. $x = 1$, we won't converge at all

Bisection

- Start with an interval $[a, b]$ where $f(a)$ and $f(b)$ are different signs (conventionally $f(a) < 0, f(b) > 0$)
- Each iteration gives an interval half as wide as before, based on checking the sign of f at the midpoint and narrowing the interval accordingly
- Error after n iterations is proportional to 2^{-n} (linear convergence), and doesn't depend much on the function or exact starting points
- A root can be found to within a specified accuracy after a predetermined number of iterations, as long as one exists in the original $[a, b]$

Illustration



Example

$f(x) = x^3 - 3x + 1$
We can take the starting interval to be $a = 1, b = 2$ since $f(1) < 0, f(2) > 0$
For the first iteration, check the midpoint, finding that $f(1.5) < 0$. Therefore, the new interval is $[1.5, 2]$
For the second iteration, find that $f(1.75) > 0$. Therefore, the new interval is $[1.5, 1.75]$
For the third iteration, find that $f(1.625) > 0$. Therefore, the new interval is $[1.5, 1.625]$
After 20 iterations, we can find the root to about 6 accurate digits $(2^{-20} \approx 10^{-6})$

Comparing and contrasting

- Newton's method
  - Requires function derivative
  - Often fast (i.e., very accurate after a few iterations) (quadratic convergence)
  - But doesn't always converge (quickly or at all)
- Bisection
  - No derivative required
  - Relatively slow, but predictable convergence rate
  - Always converges (when f is continuous)

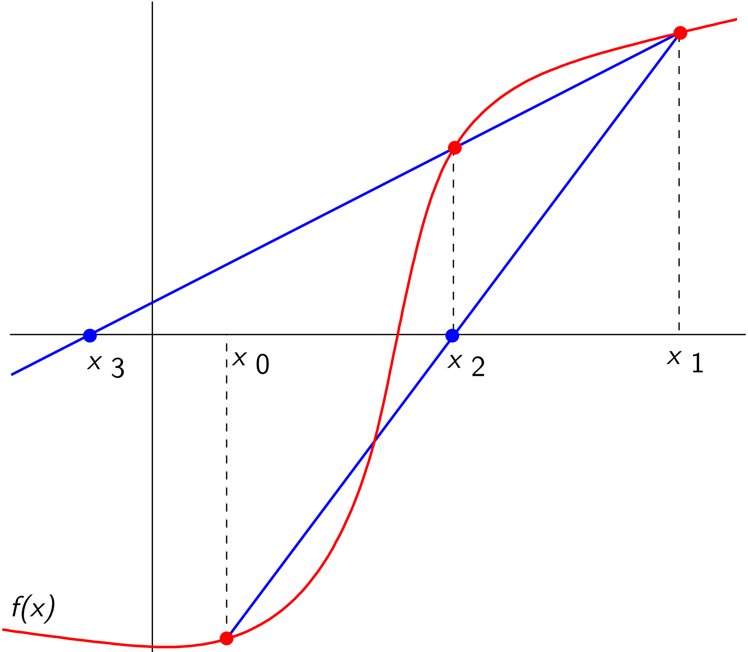
Two more, in-between methods

- Secant and false position
- Don't require derivative
- 'Usually' converge faster than bisection (but not as fast as Newton's method)
- But may not always converge (quickly)

Secant method

- Like Newton's method, but replacing $f'(x_i)$ with the slope estimated from the values of f at last 2 points
- Needs 2 initial starting points

Illustration



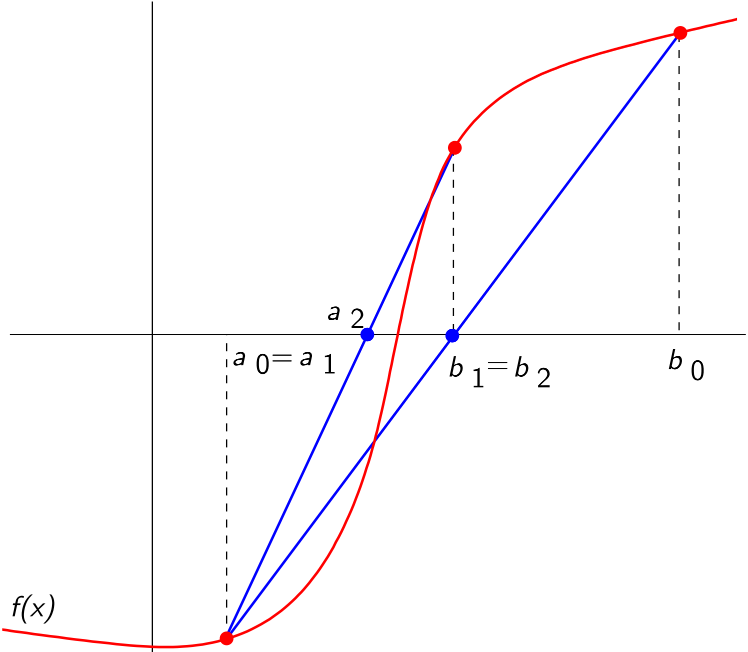
Example

$f(x) = x^3 - 3x + 1$
Starting with $a = 1, b = 2$, we get $s = 4, c = 1.25$
For the next iteration, we get $s = 5.0625, c = 1.4074$
Next, $s = 2.3026, c = 1.5961$
Note that the root isn't always in $[a, b]$

False position method

- Like bisection, but using the result from the secant method formula instead of the midpoint to be one of the interval endpoints in the next iteration
- Needs 2 initial points as in bisection

Illustration



Example

$f(x) = x^3 - 3x + 1$
Starting with $a = 1, b = 2$, we get $s = 4, c = 1.25$, and check that $f(1.25) < 0$
With $a = 1.25, b = 2$, we get $s = 5.0625, c = 1.4074$, and check that $f(c) < 0$
With $a = 1.4074, b = 2$, we get $s = 5.7956, c = 1.4824$, and check that $f(c) < 0$

Multivariate case

Here we typically are looking for the solution of a system of n equations in the same number of unknowns. Again, for nonlinear equations, there can be multiple solutions.
In standard form, $\boldsymbol{f}(\boldsymbol{x}) = 0$, or

$$
\begin{aligned}
f_1(\boldsymbol{x}) &= 0 \\
f_2(\boldsymbol{x}) &= 0 \\
&\dots \\
f_n(\boldsymbol{x}) &= 0
\end{aligned}
$$

(different from the standard form for a system of linear equations, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$)

Newton's method for systems of equations

Starting from an initial guess $\boldsymbol{x}_0$,
each iteration solves the linear system $\boldsymbol{J}(\boldsymbol{x}_i)\Delta_i = \boldsymbol{f}(\boldsymbol{x}_i)$ to find the vector of increments Δ_i, so that $\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - \Delta_i$
(can also write $\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - \boldsymbol{J}^{-1}(\boldsymbol{x}_i)\boldsymbol{f}(\boldsymbol{x}_i)$)
where $\boldsymbol{J}$ is a matrix of partial-derivative functions (the *Jacobian*), defined as $J_{i,j} = \frac{\partial f_i}{\partial x_j}$

Example

For the system of equations

$$x_1 + x_2 + x_1 x_2 = 0$$
$$x_1^2 - x_2 + 3 = 0,$$

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} x_1 + x_2 + x_1 x_2 \\ x_1^2 - x_2 + 3 \end{pmatrix}$$

and

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{pmatrix} 1 + x_2 & 1 + x_1 \\ 2x_1 & -1 \end{pmatrix}$$

If we start with $\boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, Newton's method gives

$$\boldsymbol{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix},$$

$$\boldsymbol{x}_2 = \begin{pmatrix} -1.3125 \\ 1.8750 \end{pmatrix}, \boldsymbol{x}_3 = \begin{pmatrix} -0.5579 \\ 2.7419 \end{pmatrix}, \ldots$$