4. Eigenvalues and eigenvectors

Systems of differential equations

Involve unknown functions (usu. of time or space), not just unknown scalars

Here, we'll look at some simple cases where we can find solutions using matrices and linear algebra

First-order linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where each $x_i(t)$ is an unknown function and $\mathbf{A}$ is a known matrix

- Describes a system with exponential decay or growth. Example applications: chemical reactions, epidemic spread
- General solution is $\mathbf{x}(t) = \sum_{i=1}^{n} c_i \mathbf{v}_i e^{\lambda_i t}$ where n is the number of equations and unknowns, the c_i can be any numbers, λ_i are *eigenvalues* of $\mathbf{A}$, and $\mathbf{v}_i$ are corresponding *eigenvectors*

Example

$$\text{For}\quad \begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2, \end{aligned}$$

The eigenvalues of the coefficient matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ are $-1, 3$ and corresponding eigenvectors are $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The general solution is therefore

$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$

If it's also given that $x_1(0) = 3, x_2(0) = 1$, then $c_1 = -1, c_2 = 2$, and we can write

$$\begin{aligned} x_1(t) &= e^{-t} + 2e^{3t} \\ x_2(t) &= -e^{-t} + 2e^{3t}. \end{aligned}$$

Definition of eigenvalues and eigenvectors of a matrix

A nonzero vector $\mathbf{v}$ is an eigenvector of a (square) matrix $\mathbf{A}$ if $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. λ is then the corresponding eigenvalue of $\mathbf{A}$.

- Example: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}$, with eigenvalue 5.
- Exercise: check if $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is also an eigenvector of this matrix; if so, what's its eigenvalue?

Second-order linear system

$$\mathbf{M}\mathbf{x}'' = -\mathbf{K}\mathbf{x}$$

where each $x_i(t)$ is an unknown function and $\mathbf{M}, \mathbf{K}$ are given *mass* and *stiffness* matrices respectively

- Describes a system with oscillations (e.g. pendulum, mass-spring system). Example applications: vibrations, waves, electric circuits
- These types of equations can be derived by combining Newton's second law $mx'' = F$ with Hooke's law for a spring, $F = -kx$
- General solution is $\mathbf{x}(t) = \sum_{i=1}^{n} c_i \mathbf{v}_i \sin(\sqrt{\lambda_i} t) + d_i \mathbf{v}_i \cos(\sqrt{\lambda_i} t)$ where n is the number of equations and unknowns, the c_i, d_i can be any numbers, λ_i are eigenvalues of $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$, and $\mathbf{v}_i$ are corresponding eigenvectors

Shear building model

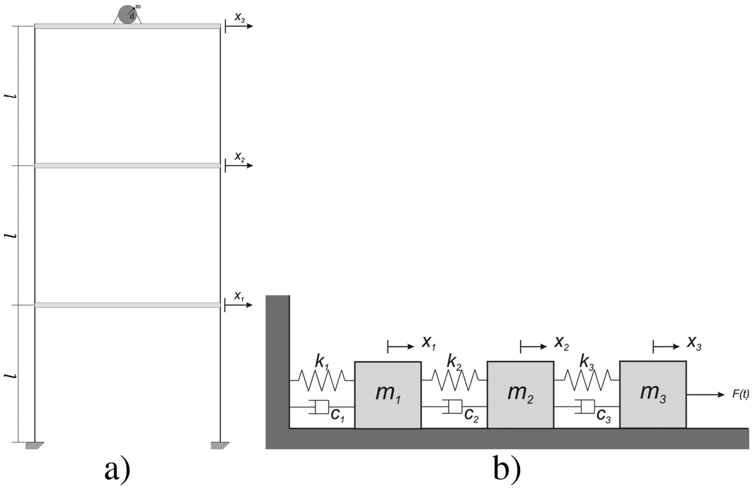

Figure: Swaying due to wind or earthquake is modeled as a system of linear oscillators

Example

$$\text{For} \quad \begin{aligned} \tfrac{1}{2}x_1'' &= -5x_1 + x_2 \\ \tfrac{1}{3}x_2'' &= -x_1 - x_2, \end{aligned}$$

The eigenvalues of the coefficient matrix $\mathbf{A} = \begin{pmatrix} 10 & -2 \\ 3 & 3 \end{pmatrix}$ are $9, 4$ and corresponding eigenvectors are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

The general solution is therefore $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin(3t) + d_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos(3t) + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \sin(2t) + d_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cos(2t)$.

If it's also given that $x_1(0) = 3, x_2(0) = 1, x_1'(0) = -1, x_2'(0) = -3$, then $c_1 = 0, c_2 = -0.5, d_1 = 1.6, d_2 = -0.2$, and we can write

$$\begin{aligned} x_1(t) &= 3.2\cos(3t) - 0.5\sin(2t) - 0.2\cos(2t) \\ x_2(t) &= 1.6\cos(3t) - 1.5\sin(2t) - 0.6\cos(2t). \end{aligned}$$

Another example

$$\text{For} \quad \begin{aligned} x_1'' &= -7x_1 + 18x_2 \\ x_2'' &= 3x_1 - 10x_2, \end{aligned}$$

The eigenvalues of the coefficient matrix $\mathbf{A} = \begin{pmatrix} 7 & -18 \\ -3 & 10 \end{pmatrix}$ are $1, 16$ and corresponding eigenvectors are $\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

The general solution is therefore $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \sin(t) + d_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cos(t) + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sin(4t) + d_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos(4t)$.

If it's also given that $x_1(0) = 3, x_2(0) = 1, x_1'(0) = -1, x_2'(0) = -3$, then $c_1 = -1.4, c_2 = 0.4, d_1 = 1, d_2 = 0$ and we can write

$$\begin{aligned} x_1(t) &= 3\cos(t) - 4.2\sin(t) + 0.8\sin(4t) \\ x_2(t) &= \cos(t) - 1.4\sin(t) - 0.4\sin(4t). \end{aligned}$$

The characteristic polynomial

One method to find eigenvalues of a given matrix is to rewrite $\mathbf{Av} = \lambda\mathbf{v}$ as $(\mathbf{A} - \lambda\mathbf{I}_n)\mathbf{v} = \mathbf{0}$. Since $\mathbf{v}$ is nonzero, the fact that multiplying it by $\mathbf{A} - \lambda\mathbf{I}_n$ yields all zeros implies that $\mathbf{A} - \lambda\mathbf{I}_n$ must be singular. Since a singular matrix has a determinant of zero, we can find λ by expanding $\det(\mathbf{A} - \lambda\mathbf{I}_n)$ as a polynomial in λ and solving for its roots. Once we find each λ, we can solve the rank-deficient linear system $(\mathbf{A} - \lambda\mathbf{I}_n)\mathbf{v} = \mathbf{0}$ to find the eigenvector $\mathbf{v}$.

Example

For $\mathbf{A} = \begin{pmatrix} 4 & 10 \\ 5 & -1 \end{pmatrix}$, the characteristic polynomial is $\det(\mathbf{A} - \lambda \mathbf{I}_2) = \begin{vmatrix} 4-\lambda & 10 \\ 5 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 50 = \lambda^2 - 3\lambda - 54 = (\lambda+6)(\lambda-9)$, with roots -6 and 9. These are therefore the eigenvalues of $\mathbf{A}$.

For the eigenvalue -6, we have $\begin{pmatrix} 10 & 10 \\ 5 & 5 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so the eigenvector is $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, or any nonzero multiple thereof.

For the eigenvalue 9, we have $\begin{pmatrix} -5 & 10 \\ 5 & -10 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so the eigenvector is $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, or any nonzero multiple thereof.

Exercise

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$$

by solving for the roots of the characteristic polynomial.

Compare your answer with the result from Python's `scipy.linalg.eig` function and explain any difference you find.

The power method

An iterative method to estimate the eigenvector with largest (in absolute value) eigenvalue for a given matrix is to take any initial vector $\mathbf{v}_0$ and keep multiplying $\mathbf{A}$ by it: $\mathbf{v}_{i+1} = \mathbf{A}\mathbf{v}_i$, optionally scaling so that the magnitude stays constant

Example

$\mathbf{A} = \begin{pmatrix} 7 & 4 \\ 2 & 5 \end{pmatrix}, \mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and then (without scaling) $\mathbf{v}_1 = \begin{pmatrix} 11 \\ 7 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 105 \\ 57 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 963 \\ 495 \end{pmatrix}$, and the sequence $\{\mathbf{v}_i\}$ will keep getting closer to a multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which is the eigenvector with eigenvalue 9.