# 8. ODE boundary value problems

# Outline

- Definition and examples
- The finite difference method

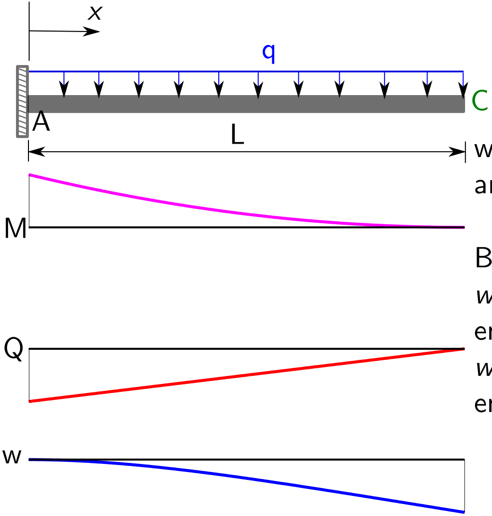
# Definition of BVPs

- An ODE of degree $n > 1$ (or system of $n > 1$ first-order ODEs) where the $n$ initial values of the unknown function $y$ (and/or its derivatives) are not all given at the same point (value of the independent variable $t$ (or $x$))
- A typical problem might be to find out the values of the unknown function within a *domain*, where the known values are given at the boundaries of the domain

# Examples of BVPs

- Beam deflection (known displacement at ends or supports)
- Heat flow (known temperatures at boundaries)
- Groundwater flow (known water levels at edges)

# Euler-Bernoulli beam equation example

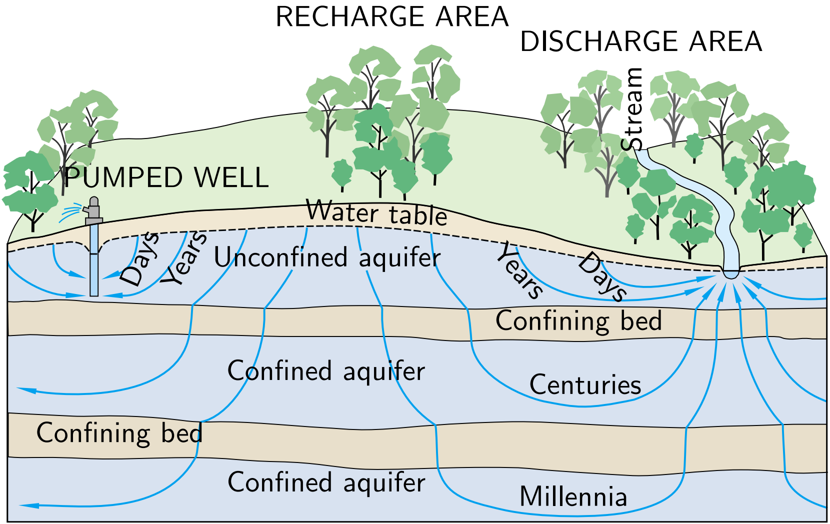



$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q$$

where $w(x)$ is displacement and $q(x)$ is loading

Boundary conditions are $w(0) = w'(0) = 0$ for built-in end
$w''(L) = w'''(L) = 0$ for free end

# Groundwater flow sketch



# A simple groundwater flow example problem

- Steady, vertically uniform 1-D flow following Darcy's Law
- ODE is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{K}\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0,$$

  where $y(x)$ is the groundwater head (water table level) and $\mathrm{K}(x)$ the hydraulic conductivity.
- Applying the derivative product rule, we can write this as

$$K\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{dK}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

- The boundary conditions are $y(x = 0) = y_0, y(x = L) = y_L$
- Take $L = 10, K = 5 + x, y_0 = 30, y_L = 20$

# The finite difference method for BVPs

Solve BVPs approximately for the unknown function values over a domain by the following steps:

1. Select a grid
2. Get an approximate system of algebraic (non-differential) equations on the grid using finite differences
3. Solve system of equations to approximate the unknown function

# 1. Select a grid

- Usually equally spaced values of the independent variable (e.g. $t$ or $x$) that cover the problem domain
- Grid should include the points where boundary values are given
- For our example: Divide the $x$ interval $[0, L]$ into $N = 5$ segments (so $N + 1$ total grid points and $N - 1$ interior points)
  - Grid points are numbered $0 = x_0, x_1, x_2, \cdots, x_N = L$; the corresponding $y$ values that will be estimated are $y_0, y_1, y_2, \cdots, y_N = y_L$

# 2. Get approximate system of equations

- Apply the differential equation at each interior grid point. Wherever that equation has a derivative, substitute a finite-difference approximation that involves grid-point values of $y$
- Add the boundary conditions, with each derivative in a boundary condition replaced by a finite-difference approximation that involves grid-point values of $y$
- For grid points at or near the edges, may need to use non-centered finite-difference approximations so that we don't go outside the chosen grid

# System of equations for our example

- Interior grid points:
  - Can use the second-order-accurate centered finite difference approximations we learned –

$$\frac{\mathrm{d}y}{\mathrm{d}x} \approx \frac{y_{i+1} - y_{i-1}}{2h}, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2},$$

    where $h$ is the grid spacing, $\frac{L-0}{N}$
  - So for our example, the algebraic equation derived for each interior grid point $i = 1, 2, \cdots N-1$ is

$$K(x_i)\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + K'(x_i)\frac{y_{i+1} - y_{i-1}}{2h} = 0$$

$$\text{or} \quad \left(\frac{K_i}{h^2} - \frac{K_i'}{2h}\right) y_{i-1} - \frac{2K_i}{h^2} y_i + \left(\frac{K_i}{h^2} + \frac{K_i'}{2h}\right) y_{i+1} = 0$$

    where $K_i, K_i'$ are the known values at the respective $x_i$, and $y_i$ are unknown values
- The boundary conditions here are simply $y_0 = 30, y_N = 20$

# Linear system in matrix form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{2} & -\frac{7}{2} & 2 & 0 & 0 & 0 \\
0 & 2 & -\frac{9}{2} & \frac{5}{2} & 0 & 0 \\
0 & 0 & \frac{5}{2} & -\frac{11}{2} & 3 & 0 \\
0 & 0 & 0 & 3 & -\frac{13}{2} & \frac{7}{2} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}
=
\begin{pmatrix} 30 \\ 0 \\ 0 \\ 0 \\ 0 \\ 20 \end{pmatrix}
$$

Most coefficients are zero (in fact, as set up here, the coefficient matrix is tridiagonal), which makes the solution easier to compute

# 3. Solve system to approximate the unknown function

- For the cases we'll consider, the differential equation is linear in $y$ and its derivatives, so the approximate system of algebraic equations is also linear in the $y_i$
- Can solve using Gauss elimination
- Result is approximate values of $y$ at each $x_i$

# Accuracy of finite differences

- The finer our grid (with a larger number $N$ of divisions), the smaller the step size $h$ in the finite difference approximations of the derivatives and therefore the more accurately we approach the actual solution to the differential equation (since the error in the approximation is proportional to, say, $h^2$)
- But then more equations need to be set up and solved (the usual tradeoff)
- In practice, we could solve for a few different values of $N$ to get confidence that our approximate finite-difference solution is accurate enough

# Result for the example