Solution to Problem 4.15

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Here, $R(x) = 1$, $P(x) = x - 1$, $Q(x) = -\alpha x$.
We therefore have

$$f(s+k) = (s+k)(s+k-1) - (s+k) = (s+k)(s+k-2) \tag{1}$$

and

$$g_n(s+k) = (s+k-1) - \alpha \text{ if } n = 1 \text{ and } 0 \text{ if } n > 1. \tag{2}$$

The indicial equation is

$$f(s) = (s)(s-2) = 0, \tag{3}$$

so that $s = 0$ or 2.
For $k \geq 1$, we therefore have

$$(s+k)(s+k-2)A_k = -((s+k-1) - \alpha)A_{k-1}. \tag{4}$$

Suppose $s = 2$. Then

$$k(k+2)A_k = -(k+1-\alpha)A_{k-1}. \tag{5}$$

Or, since $k(k+2)$ is positive for all k,

$$A_k = \frac{\alpha - k - 1}{k(k+2)} A_{k-1}. \tag{6}$$

This uniquely determines all A_k (with only A_0 arbitrary).

If α is an integer ≥ 2, the series only has finitely many terms, and the solution is a polynomial of degree α

Suppose $s = 0$. Then

$$k(k-2)A_k = -(k-1-\alpha)A_{k-1}. \tag{7}$$

For $k = 1$,

$$-A_1 = \alpha A_0. \tag{8}$$

For $k = 2$,

$$0 = (1-\alpha)A_1. \tag{9}$$

For $k \geq 3$, $k(k-2)$ is positive, and

$$A_k = \frac{\alpha - k + 1}{k(k-2)} A_{k-1}. \tag{10}$$

If $\alpha \neq 0$ or 1, we have that $A_1 = 0$, $A_0 = 0$, A_2 is arbitrary, and all subsequent A_k. But this contradicts our assumption that $A_0 \neq 0$. Therefore, there is no regular solution with $s = 0$. (The power series starting with A_2 is the same as the solution with $s = 2$.)

If $\alpha = 0$, then A_0 is arbitrary, $A_1 = 0$, and then A_2 is arbitrary (and multiplies the second solution). So the general solution is

$$A_0 + A_2 x^2(1 - 2x/3 + x^2/4 - x^3/15 + \ldots) = A_0 + 2A_2 x^2(1/2! - 2x/3! + 3x^2/4! - 4x^3/5! + \ldots). \tag{11}$$

If $\alpha = 1$, then A_0 is arbitrary, $A_1 = -A_0$, and then A_2 is arbitrary again. So the general solution is

$$A_0(1-x) + A_2 x^2(1 - x/3 + x^2/12 - x^3/60 + \ldots) = A_0(1-x) + 2A_2 x^2(1/2! - x/3! + x^2/4! - x^3/5! + \ldots). \tag{12}$$