Exam 1 solutions

(a)

$$\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=\frac{h(t)}{m}$$

(b)
This is a linear second-order ordinary differential equation with constant coefficients.

(c)
We assume a solution of the form $x = Ce^{rt}$, giving for r the quadratic equation:
$r^2 + (c/m)r + (k/m) = 0$
Using the quadratic formula:

$$r=\frac{(-c\pm\sqrt{(c^2-4km)})}{2m} \quad .$$

If these roots are distinct, the general solution is $x=C_1e^{r_1t}+C_2e^{r_2t}$. If $c=2\sqrt{km}$, there is only one root and the general solution is $x=C_1e^{-ct/2\mathrm{m}}+C_2te^{-ct/2\mathrm{m}}$.

(d)
For distinct roots, we have $C_1 + C_2 = b$ and $r_1C_1 + r_2C_2 = 0$. So $C_1 = -(r_2/r_1)C_2 = b - C_2 \rightarrow (1 - r_2/r_1)C_2 = b \rightarrow C_2 = b / (1 - r_2/r_1), C_1 = b/(1 - r_1/r_2)$.

If there is only one root, the condition on $x(0)$ gives $C_1 = b$ and the condition on $x'(0)$ gives $C_2 = bc/2m$.

(e)
If $c>2\sqrt{km}$, there are two negative real roots, so the solution is the combination of two exponential decay curves. If $c=2\sqrt{km}$, the solution also decays but has a component that is proportional to t times an exponential decay. If $c<2\sqrt{km}$, the roots are complex (with negative real part), and the solution has oscillations (sine/cosine waves) superimposed on an exponential decay.

(f)
$h(t)$ and its derivatives has a one-member family of basis functions, $\{e^{-t}\}$. So we look for a solution of the form Ce^{-t} . Substituting this in the differential equation, we find that $x(t)=\dfrac{e^{-t}}{m-c+k}$ is a solution. ($(m - c + k) = 0$ if and only if e^{-t} is a homogenous solution.)

(g)
Laplace transform of each term in the equation yields

$$ms^2\bar{x}(s)-msx(0)-mx'(0)+cs\bar{x}(s)-x(0)+k\bar{x}(s)=a \quad .$$

Putting in the given initial conditions,

$$ms^2\bar{x}(s)+cs\bar{x}(s)+k\bar{x}(s)=a$$

or

$$\bar{x}(s)=\frac{a}{ms^2+cs+k} \quad .$$