1]

(a) If a solution is in the form $y(x)=\sum_{n=0}^{\infty} A_n x^n$, then substituting into the equation gives us

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)A_n x^{n-2}-2\mathrm{x}\sum_{n=0}^{\infty} nA_n x^{n-1}+2\sum_{n=0}^{\infty} A_n x^n=0$$, or

$$\sum_{n=2}^{\infty} n(n-1)A_n x^{n-2}+\sum_{n=0}^{\infty}(2-2\mathrm{n}-n(n-1))A_n x^n=0$$, or

$$\sum_{n=0}^{\infty}(n+2)(n+1)A_{n+2}x^n+(2-2\mathrm{n}-n(n-1))A_n x^n=0$$.

So the condition on the coefficients A_n (n from 0 to ∞) is

$$(n+2)(n+1)A_{n+2}+(2-2\mathrm{n}-n(n-1))A_n=0$$, or

$$(n+2)(n+1)A_{n+2}-(n+2)(n-1)A_n=0$$.

This, A_0 and A_1 are arbitrary, all even A_n are determined by A_0 , and all odd A_n are determined by A_1 . The first independent solution is:

$$A_0(1-x^2-x^4/3-x^6/5-x^8/7-...)$$.

The second independent solution is just

$A_1 x$ (all the higher odd coefficients are zero).

The general solution therefore is

$$y(x)=c_1(1-x^2-x^4/3-x^6/5-x^8/7-...)+c_2 x$$.

(b) The first independent solution has a convergence radius of 1 (converges for x in (-1, 1)), as given by the ratio test.
The second independent solution converges for all x, because it is a polynomial in x.

(c) This is Legendre's equation with $p = 1$.

2]

(a) The general solution is

$$y(x)=c_1\sin(\sqrt{\lambda}x)+c_2\cos(\sqrt{\lambda}x)$$, so

$$y'(x)=\sqrt{\lambda}(c_1\cos(\sqrt{\lambda}x)-c_2\sin(\sqrt{\lambda}x))$$.

Given the boundary conditions, $c_1=0$, and nonzero c_2 is possible only if $\sqrt{\lambda}$ is an integer, so that $\sin(\sqrt{\lambda}\pi)=0$.

So the characteristic numbers are

$\lambda_n=n^2$ where n = 0, 1, 2, 3, ...

and the corresponding characteristic functions are

$$\phi_n(x)=\cos(nx)$$.

(b)

Since the right-hand side is one of the characteristic functions while 3 is not one of the characteristic values (so there are no solutions to the corresponding homogenous boundary-value problem), Eq. 124 in Ch. 5 gives

$$y(x)=\frac{1}{3-16}\cos(4x)$$

which can be verified by substituting into the differential equation.

3] The general formula for a Fourier series on $[-\pi,\pi]$ is:

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du+\frac{1}{\pi}\sum_{n=0}^{\infty}\left[\int_{-\pi}^{\pi} f(u)\cos(nu)\,du\right]\cos(nx)+\frac{1}{\pi}\sum_{n=0}^{\infty}\left[\int_{-\pi}^{\pi} f(u)\sin(nu)\,du\right]\sin(nx)\quad .$$

In this case, we have 1/2 for the average value of the function;

$$\int_{-\pi}^{\pi} f(u)\cos(nu)\,du=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos(nu)\,du=\frac{1}{n}(\sin(n\pi/2)-\sin(-n\pi/2))=\frac{2}{n}\sin(n\pi/2)$$, where

$\sin(n\pi/2)$ is 0 for even *n* and $(-1)^{(n-1)/2}$ (alternating 1 and -1) for odd *n*;

while, since $f(x)$ is even and $\sin(x)$ is odd, $\int_{-\pi}^{\pi} f(u)\sin(nu)\,du=0$.

We therefore obtain

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\left(\cos(x)-\frac{\cos(3x)}{3}+\frac{\cos(5x)}{5}-\frac{\cos(7x)}{7}+...\right)\quad .$$

Because $f(x)$ is periodic with period 2π , this series converges to $f(x)$ not only in $(-\pi,\pi)$ but over the entire real line.