Solutions to Final Exam

May 24, 2011

problem 1

a

At steady state, we have

$$
\begin{aligned}
&T_{xx} + C = 0 \text{ , for } 0 < x < L \\
&T(x=0) = 0, T_x(x=L) = 0
\end{aligned}
$$

The general solution to the ODE is

$$T = \frac{-C}{2}x^2 + ax + b$$

The boundary conditions give us $b = 0$, $a = CL$, so we finally have

$$T_S(x) = Cx(L - \frac{x}{2})$$

which plots as the left half of a down-facing parabola. Note that the hottest section of the rod, at $x = L$, is at a temperature of $CL^2/2$.

b

Defining $T_T \equiv T - T_S$, where we have T_S from (a), gives for T_T (referred to as just T for simplicity)

$$
\begin{aligned}
&T_{xx} = \frac{1}{\alpha^2}T_t \text{ , for } 0 < x < L, t > 0 \\
&T(x=0,t) = 0, T_x(x=L,t) = 0, T(x,t=0) = f^*(x) \equiv f(x) - Cx(L - \frac{x}{2}), \lim_{t\to\infty} T(x,t) = 0
\end{aligned}
$$

c

Assuming solutions of the form $X(x)\Theta(t)$, we have from the PDE

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{\Theta'}{\Theta} \equiv -k^2$$

giving the general solutions

$$X'' + k^2 X = 0 \to X = c_1 \sin kx + c_2 \cos(kx)$$
$$\Theta' + \alpha^2 k^2 \Theta = 0 \to \Theta = de^{-k^2\alpha^2 t}$$

The boundary conditions at the ends of the rod give us a Storm-Liouville problem, with characteristic functions

$$X_n = \sin k_n x,\ k_n \equiv \frac{(n-\frac{1}{2})\pi}{L},\ n = 1, 2, 3, \ldots$$

By superposing these solutions we get the transient temperature distribution

$$T(x,t) = \sum_{n=1}^{\infty} a_n \sin(k_n x) e^{-k_n^2 \alpha^2 t}$$

where a_n can be determined from the orthogonal function expansion of the initial condition:

$$a_n = \frac{2}{L}\int_0^L f^*(x)\sin(k_n x)\mathrm{d}x$$

The timescale for the exponential decay of each transient mode is given by

$$\frac{1}{k_n^2\alpha^2}.$$

The longest timescale corresponds to the smallest k, namely k_1, and is equal to

$$\frac{4L^2}{\pi^2\alpha^2}.$$

d

As long as C is not changing with time, only the steady-state part of the solution is affected. This becomes

$$T_S(x) = ax + b - \int_0^x \int_0^\chi C(\xi)\mathrm{d}\xi\mathrm{d}\chi,$$

where the boundary conditions give $b = 0$, and

$$a = \int_0^L C(x)\mathrm{d}x = L\bar{C}$$

, where $\bar{C}$ is the average heat production rate over $0 < x < L$. If $C(x) = x$, we have $\bar{C} = L/2$, and

$$T_S(x) = \frac{x}{2}(L^2 - \frac{x^2}{3}).$$

T_T is as before, with modified $f^* = T - T_S$.

problem 2

a-b

Assuming solutions of the form $R(r)\Theta(\theta)$, we have from the PDE

$$r^2\frac{R''}{R} + r\frac{R'}{R} = \frac{\Theta''}{\Theta} \equiv -k^2.$$

or

$$\Theta'' + k^2\Theta = 0 \to \Theta = c_1 \sin k\theta + c_2 \cos(k\theta)$$
$$r^2R'' + rR' + k^2R = 0 \to R = d_1 r^{-k} + d_2 r^k.$$

c

The conditions at $\theta = 0$ and $\theta = \pi$ give a Storm-Liouville problem for Θ, with characteristic functions

$$\Theta_n = \sin n\theta,\ n = 1, 2, 3, \ldots$$

Given that k must be positive integers, the requirement that the field be bounded at large r gives $d_2 = 0$.

So solutions are of the form

$$T_n = r^{-n} \sin n\theta,\ n = 1, 2, 3, \ldots$$

d

We take

$$T(r, \theta) = \sum_{n=1}^{\infty} a_n r^{-n} \sin n\theta$$

with coefficients determined by the boundary conditions at $r = 1$:

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin(n\theta)\mathrm{d}\theta.$$