# Solutions to Homework 11

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## 9.52

The problem is

$$w_{xx} + w_{yy} = \frac{1}{c^2} w_{tt} \text{ , for } 0 < x < L, 0 < y < L$$
$$w(x = 0) = w(x = L) = w(y = 0) = w(y = L) = 0; w \text{ bounded.}$$

Looking for separable solutions by substituting

$$w(x, y, t) = X(x)Y(y)T(t),$$

we get

$$X''YT + XY''T = \frac{1}{c^2} XYT'' \text{ , for } 0 < x < l, 0 < y < l$$
$$X(0) = X(L) = Y(0) = Y(L) = 0; X, Y, T \text{ bounded.}$$

or

$$\frac{X''}{X} = \frac{T''}{c^2 T} - \frac{Y''}{Y} = -k^2$$
$$\frac{T''}{c^2 T} + k^2 = \frac{Y''}{Y} = -l^2$$

giving us Storm-Liouville problems for $X$ and $Y$, with general solutions

$$X'' + k^2 X = 0 \rightarrow X = c_1 \sin(kx) + c_2 \cos(kx)$$
$$Y'' + l^2 Y = 0 \rightarrow Y = d_1 \sin(ly) + d_2 \cos(ly)$$

Using the boundary conditions, we need

$$c_2 = 0,\ k_m = \frac{m\pi}{L},\ m = 1, 2, 3, \ldots$$
$$d_2 = 0,\ l_n = \frac{m\pi}{L},\ n = 1, 2, 3, \ldots$$

For $T$, the general solution is

$$T'' + c^2(k^2 + l^2)T = 0 \to T = e_1 \sin(c\sqrt{k^2 + l^2}t) + e_2 \cos(c\sqrt{k^2 + l^2}t)$$

with no boundary conditions applicable. However, the requirements on $k$ and $l$ mean that $c\sqrt{k^2 + l^2}$ will be in the form $c\sqrt{\frac{m^2\pi^2}{L^2} + \frac{m^2\pi^2}{L^2}}$, or $\frac{c\pi}{L}\sqrt{m^2 + n^2}$. Considering a superposition of these separable solutions across positive integer $m$ and $n$ and consolidating the coefficients into $a_{mn} \equiv c_{1m}d_{1n}e_{2mn}$, $b_{mn} \equiv c_{1m}d_{1n}e_{1mn}$, we get

$$w(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi y}{L})(a_{mn} \cos(\sqrt{m^2 + n^2}\frac{c\pi t}{L} + b_{mn} \sin(\sqrt{m^2 + n^2}\frac{c\pi t}{L}))$$

as requested (noting that $c^2 = \frac{T}{\rho}$).

## 9.58

(a) The problem for the steady-state case is

$$\begin{aligned} &T_{xx} = 0 \\ &T(0) = T_1,\ hlT_x(L) + T(L) - T_0 = 0 \end{aligned}$$

The general solution is $T(x) = a + bT$, and putting in the boundary conditions we get

$$a = T_1, b = \frac{T_0 - a}{L + hL}$$

, leading to

$$T_S(x) = T_1 - \frac{T_1 - T_0}{1 + h}\frac{x}{L}.$$

(b) The problem for the transient distribution $T_T(x, t)$ (henceforth just $T$) now is

$$\begin{aligned} &T_{xx} = \frac{1}{\alpha^2}T_t,\ 0 < x < L,\ t > 0 \\ &T(0, t) = 0,\ hlT_x(L, t) + T(L, t) = 0,\ T(x, 0) = f(x) - T_S(x),\ \lim_{t\to\infty} T = 0 \end{aligned}$$

Looking for separable solutions,

$$\begin{aligned} &T = X(x)\mathrm{T}(t) \to \\ &X''\mathrm{T} = \frac{1}{\alpha^2}X\mathrm{T}' \to \\ &\frac{X''}{X} = \frac{\mathrm{T}'}{\alpha^2 T} = -k^2 \\ &X(0) = 0, hLX'(L) + X(L) = 0,\ \lim_{t\to\infty} \mathrm{T} = 0 \end{aligned}$$

giving us general solutions for $X$ and T

$$
\begin{aligned}
&X'' + k^2 X = 0 \to X = c_1 \sin(kx) + c_2 \cos(kx) \\
&\mathrm{T}' + k^2\alpha^2 \mathrm{T} = 0 \to T = d_1 e^{-k^2\alpha^2 t}
\end{aligned}
$$

From the homogenous boundary conditions on $X$, $c_2 = 0$ and $hLk\cos(kL) + \sin(kL) = 0$, implying that $kL$ must be a root of the equation $hx + \tan(x) = 0$. Call these roots $k_n$.

The homogenous boundary condition for T is satisfied as long as $k^2$ is positive.

The general solution is obtained by superposition over $n$:

$$
\sum_{n=1}^{\infty} a_n \sin \frac{k_n x}{L} e^{-k_n{}^2\alpha^2 t/L^2}
$$

with $a_n \equiv c_{1n} d_{1n}$.

(c) Because $X_n(x)$ satisfy a Storm-Liouville problem, the characteristic functions $\{\sin \frac{k_n x}{L}, n = 1, 2, 3, \ldots\}$ form an orthogonal set over $[0, L]$ and the function $f(x) - T_S(x)$ can be expanded in terms of them, with expansion coefficients given by

$$
A_n = \frac{\int_0^L (f(x) - T_S(x)) \sin \frac{k_n x}{L} \mathrm{d}x}{\int_0^L \sin^2 \frac{k_n x}{L} \mathrm{d}x}
$$

and equating these with $a_n$ in the general solution to the PDE we obtain a solution that meets all the boundary conditions.

## 9.62

(a)

The problem is

$$
\begin{aligned}
&T_{rr} + \frac{1}{r} T_r = \frac{1}{\alpha^2} T_t,\ 0 \le r < a,\ t > 0 \\
&T(a, t) = T_0 \cos(\omega t),\ T \text{ finite}
\end{aligned}
$$

For the proposed solution, we have

$$
T(r,t) = \mathfrak{Re}(F(r)e^{i\omega t}) = \mathfrak{Re}(F(r))\,\mathfrak{Re}(e^{i\omega t}) - \mathfrak{Im}(F(r))\,\mathfrak{Im}(e^{i\omega t}) = \cos(\omega t) F^R(r) + \sin(\omega t) F^I(r)
$$

where $F^R \equiv \mathfrak{Re}(F)$ and $F^I \equiv \mathfrak{Im}(F)$. The condition at $r = a$ gives that $\cos(\omega t) F^R(a) + \sin(\omega t) F^I(a) = T_0 \cos(\omega t)$, which requires $F^I(a) = 0$ and $F^R(a) = T_0$. Also, if $F^R(0)$ or $F^I(0)$ are infinite than $T(0, t)$ will be infinite at $t = 0$ or $t = \frac{\pi}{2\omega}$ respectively, so both must be finite. Further,

$$
\begin{aligned}
T_{rr} &= \cos(\omega t)F^R_{rr} + \sin(\omega t)F^I_{rr} = \mathfrak{Re}(F_{rr}e^{i\omega t})\\
T_r &= \cos(\omega t)F^R_r + \sin(\omega t)F^I_r = \mathfrak{Re}(F_r e^{i\omega t})\\
T_t &= -\omega\sin(\omega t)F^R + \omega\cos(\omega t)F^I = \omega\,\mathfrak{Re}(iFe^{i\omega t})
\end{aligned}
$$

Substituting into the PDE, assuming that the imaginary part also satisfies it so that we can remove the $\mathfrak{Re}()$, and dividing by the common factor $e^{i\omega t}$, we get the given equation.

(b)

The equation derived has the form of Eq. 4.145, with solutions $Z_0(i^{3/2}\frac{\sqrt{\omega}}{\alpha}r)$, where the most general solution that is finite when $r = 0$ is $F(r) = CJ_0(i^{3/2}\frac{\sqrt{\omega}}{\alpha}r) = C(\text{ber}(\frac{\sqrt{\omega}}{\alpha}r)+i\text{bei}(\frac{\sqrt{\omega}}{\alpha}r))$. Adding the boundary condition at $r = a$ to determine $C$ gives the form shown.

(c) From above,

$$
T(r,t) = \qquad \mathfrak{Re}(U(r,t)) = T_0\,\mathfrak{Re}\{\frac{\text{ber}(kr)+i\text{bei}(kr)}{\text{ber}(ka)+i\text{bei}(ka)}(\cos(\omega t)+i\sin(\omega t)\}
$$

Now a complex number $z = x + iy$ can be written as $\sqrt{x^2+y^2}e^{i\tan^{-1}y/x}$ (magnitude times phase). So

$$
\text{ber}(kr)+i\text{bei}(kr) = \sqrt{\text{ber}^2(kr)+\text{bei}^2(kr)}e^{i\tan^{-1}\frac{bei(kr)}{ber(kr)}} = M_0(kr)e^{i\theta_0(kr)}
$$

, where $M_0$ and $\theta_0$ are real. Substituting,

$$
\begin{aligned}
&T(r,t) = \mathfrak{Re}(U(r,t)) = \mathfrak{Re}(\frac{M_0(kr)}{M_0(ar)}e^{i(\theta_0(kr)-\theta_0(ar)+\omega t)})\\
&= \frac{M_0(kr)}{M_0(ar)}\mathfrak{Re}((\cos(\theta_0(kr)-\theta_0(ar)+\omega t)+i\sin(\theta_0(kr)-\theta_0(ar)+\omega t))\\
&= \frac{M_0(kr)}{M_0(ar)}\cos(\theta_0(kr)-\theta_0(ar)+\omega t)
\end{aligned}
$$

(d)

$\text{ber}(0) = 1$ and $\text{bei}(0) = 0$, so $M_0(0) = 1$ and $\theta_0(0) = 0$. Substituting into the above, we get the given expression.

## 9.72

(a)

This follows because the general solutions from separation of variables are

$$X(x) = c_1 \sin(kx) + c_2 \cos(kx)$$
$$T(t) = d_1 \sin(kct) + d_2 \cos(kct)$$

and the homogenous boundary conditions give $c_2 = 0, d_1 = 0$ and $k_n = n\pi x/L, n = 1, 2, 3, \ldots$. This is a Storm-Liouville problem in $X(x)$, and the formula for the coefficients $a_n$ follows from the Fourier sine series for $f(x)$ setting $t = 0$.

(b) $2(\sin(n\pi x/L)\cos(n\pi ct/L)) = \sin(n\pi x/L+n\pi ct/L)+\sin(n\pi x/L-n\pi ct/L)$ by the trigonometric identity $2\sin A\cos B = \sin(A+B)\sin(A-B)$, so the forms are term-by-term identical.

## 9.73

(a)

Separation of variables gives us

$$\begin{aligned}
&c^2X''T = XT'' + 2\gamma XT' \\
&\to \frac{X''}{X} = \frac{T''}{c^2T} + 2\gamma\frac{T'}{c^2T} = -k^2 \\
&\to X'' + k^2X = 0, T'' + 2\gamma T' + k^2c^2T = 0
\end{aligned}$$

where the exponents for the exponential solutions are $\pm ikx$ for $X(x)$ and

$$(-\gamma \pm \sqrt{\gamma^2 - k^2c^2})t = (-\gamma \pm ick\sqrt{1 - \frac{\gamma^2}{k^2c^2}})t$$

for $T(t)$. Combining the two gives us the form shown.

(b)

Expressing $X(x)$ in sines and cosines, the boundary conditions eliminate the cosine term and set $k_n = \frac{n\pi}{L}, n = 1, 2, 3, \ldots$. Expressing $T(t)$ as $e^{-\gamma t}$ times sines and cosines in $\sqrt{k_n^2c^2 - \gamma^2}t$ gives solutions $T(t) = e^{-\gamma t}(c_1\cos(\omega_n t) + c_2\sin(\omega_n t))$, where $\omega_n \equiv \sqrt{k_n^2c^2 - \gamma^2}$. The homogenous boundary condition $T'(0) = 0$ gives $-\gamma c_1 + \omega_n c_2 = 0$. The coefficients $a_n$ are obtained from equating $w(x, 0)$ with the Fourier sine series of $f(x)$.