EXTENDING THE BLENDED GENERALIZED EXTREME VALUE DISTRIBUTION

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ABSTRACT. The generalized extreme value (GEV) distribution is commonly employed to help estimate the likelihood of extreme events in many geophysical and other application areas. The recently proposed blended generalized extreme value (bGEV) distribution modifies the GEV with positive shape parameter to avoid a hard lower bound that complicates fitting and inference. Here, the bGEV is extended to the GEV with negative shape parameter, avoiding a hard upper bound that is unrealistic in many applications. This extended bGEV is shown to improve on the GEV for forecasting future heat extremes based on past data. Software implementing this bGEV and applying it to the example temperature data is provided.

1. INTRODUCTION

The generalized extreme value (GEV) distribution can be derived as the limit distribution of block maxima (from any distribution for which such a limit exists) for sufficiently large blocks and numerous samples (the Fisher–Tippett–Gnedenko theorem) [1]. However, in practice the GEV distribution is fitted to finite data sets, such as observations of temperature, precipitation, streamflow, and sea level (among many other applications to problems in fields ranging from finance to biology), to estimate the likelihood of future extreme events [2, 3]. In this context, some properties of the GEV distribution complicate inference.

The GEV is a function of 3 parameters, a real shape parameter ξ , real location parameter μ , and positive scale parameter σ . In terms of standardized values $s = \frac{x-\mu}{\sigma}$, its cumulative distribution function can be written as

(1)
$$F_{\text{GEV}}(s) = \begin{cases} \exp(-e^{-s}) & \text{if } \xi = 0\\ \exp\left(-(1+\xi s)^{-\frac{1}{\xi}}\right) & \text{if } \xi \neq 0 \text{ and } \xi s > -1\\ 0 & \text{if } \xi > 0 \text{ and } \xi s \leq -1\\ 1 & \text{if } \xi < 0 \text{ and } \xi s \leq -1. \end{cases}$$

The GEV distribution thus has positive density for all real x only in the Gumbel case, when $\xi = 0$. In the Fréchet case, when $\xi > 0$, there is zero probability that $\frac{x-\mu}{\sigma} \leq -\frac{1}{\xi}$. In the Weibull case, when $\xi < 0$, there is zero probability that $\frac{x-\mu}{\sigma} \geq -\frac{1}{\xi}$. This property does not match well the behavior of finite sets of block maxima derived, for example, from geophysical phenomena, which typically do not have an intrinsic sharp upper or lower bound. While admissible GEV parameters can be found for any collection of block maxima, forecasting with any such distribution may lead to surprises where a new record value had zero probability. As a result, a GEV distribution would be arbitrarily bad when assessed using the expected likelihood of new observations [4, 5]. Moreover, some common computational tools for inference with data,

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such as automatic differentiation variational inference [6], require the statistical distributions used to have unbounded support, and so cannot be easily employed with the GEV distribution.

In order to overcome such problems, [7] suggested a blended generalized extreme value (bGEV) distribution as a blend of the GEV distribution with $\xi > 0$ and the Gumbel distribution (i.e. GEV with $\xi = 0$), where the blended distribution would be identical to the Fréchet case over most of its probability mass but still enjoy unbounded support. Their bGEV cumulative distribution function can be written as

(2)
$$F_{\text{bGEV}}(s) = F_{\text{GEV}}(s)^{p(s)} F_{\text{Gumbel}}(\tilde{s})^{1-p(s)}.$$

 $F_{\text{GEV}}(s)$ is the GEV distribution at $s = \frac{x-\mu}{\sigma}$ with the chosen parameters $\xi > 0, \mu, \sigma$. $F_{\text{Gumbel}}(\tilde{s})$ is the Gumbel distribution with location and scale parameters $\tilde{\mu}, \tilde{\sigma}$ used to standardize $\tilde{s} = \frac{x-\tilde{\mu}}{\tilde{\sigma}}$. $\tilde{\mu}, \tilde{\sigma}$ are chosen such that $F_{\text{Gumbel}}(\tilde{s}_a) = F_{\text{GEV}}(s_a) = a$ and $F_{\text{Gumbel}}(\tilde{s}_b) = F_{\text{GEV}}(s_b) = b$ for two user-specified quantiles 0 < a < b < 1. p(s) is a function that equals 0 for $s \leq s_a$, 1 for $s \geq s_b$, and, for intermediate values $s_a < s < s_b$, is equal to the cumulative distribution function of the beta distribution $F_{\text{Beta}}(\frac{s-s_a}{s_b-s_a};\alpha,\beta)$ with user-specified positive shape parameters α, β . [7] recommend $a = 0.05, b = 0.2, \alpha = \beta = 5$.

This bGEV distribution retains the fatter right tail of the GEV Fréchet case, while having the Gumbel distribution's unbounded left tail. [7] show an example application to monthly maximum air pollutant concentrations, and the bGEV distribution has also been applied to annual maximum precipitation intensities [8] and to currency exchange rates [9]. However, whereas precipitation extremes typically have tail behavior consistent with positive ξ [10], other types of meteorological extremes such as temperature [11, 12] and sea level [13] instead have shorter tails consistent with negative ξ . Fitting small numbers of samples to the GEV distribution with negative ξ means that the fitted distribution has a hard upper bound and therefore critically underestimates the occurrence of, e.g., new heat extremes [14].

To address this problem, this note proposes a simple extension of the bGEV to the $\xi < 0$ case, and applies it to example temperature data. The principle of blending the GEV with the Gumbel distribution at its hard limit so that the support is unbounded is the same, and the details are closely analogous to those derived by [7] for $\xi > 0$, except for that the blending takes place at the upper rather than the lower tail.

Precisely, for negative ξ , the proposed bGEV has cumulative distribution function

(3)
$$F_{\text{bGEV}}(s) = F_{\text{GEV}}(s)^{p(s)} F_{\text{Gumbel}}(\tilde{s})^{1-p(s)}.$$

 $F_{\text{GEV}}(s)$ is the GEV distribution at $s = \frac{x-\mu}{\sigma}$ with the chosen parameters $\xi < 0, \mu, \sigma$. $F_{\text{Gumbel}}(\tilde{s})$ is the Gumbel distribution with location and scale parameters $\tilde{\mu}, \tilde{\sigma}$ used to standardize $\tilde{s} = \frac{x-\tilde{\mu}}{\tilde{\sigma}}$. $\tilde{\mu}, \tilde{\sigma}$ are chosen such that $F_{\text{Gumbel}}(\tilde{s}_a) = F_{\text{GEV}}(s_a) = a$ and $F_{\text{Gumbel}}(\tilde{s}_b) = F_{\text{GEV}}(s_b) = b$ for two user-specified quantiles 1 > a > b > 0. In this case a, b will typically be closer to 1 so modification of the Weibull GEV distribution is concentrated near its problematic upper bound, as opposed to the $\xi > 0$ case where a, b will typically be chosen to be closer to 0. p(s) is a function that equals 0 for $s \ge s_a$, 1 for $s \le s_b$, and, for intermediate values $s_b < s < s_a$, is equal to the cumulative distribution function of the beta distribution $F_{\text{beta}}(\frac{s-s_a}{s_b-s_a};\alpha,\beta)$ with user-specified positive shape parameters α, β . The analytic expressions for the probability density function of the bGEV given by [7] are also readily extensible. Finally, the limit of the bGEV distribution as ξ approaches 0 from either direction is simply the Gumbel distribution.

2. Example application

Recent years have seen long-standing temperature records in many countries broken by unprecedented amounts. The state-of-the-art ERA5 reanalysis [15, 16] shows this pattern well, with most of the globe reaching new temperature records in the last 15 years. As a preliminary evaluation of the ability of GEV and bGEV distributions to fit this type of data, a sample was taken of 100 populated land grid cells spaced at least 10° apart, sufficient for good global coverage. For each chosen grid cell, annual maximum temperatures for 1940-2023, in units of K, were determined from the ERA5 hourly 2-m air temperature series, resulting in a time series of n = 84 block maxima.

The GEV and bGEV distributions were compared in terms of one-year-ahead probabilistic forecasts, using negative log likelihood (NLL) of the actual value in the forecast as the skill metric [17, 5, 18]. The probabilistic forecasts were generated by fitting, for each length s from 30 up to n - 1, GEV and bGEV distributions for the first s annual maxima, and using those to generate a forecast for year s + 1 that could be compared to the observed value. NLL was therefore summed over 5400 separate forecasts (54 years for each of 100 locations).

In order to capture the observed global warming trend, the location parameter in the GEV distribution was assumed to be a linear function of the mean annual global temperature (also calculated from ERA5), i.e., for each location, $\mu(t) = \mu_0 + \mu_t(T(t) - \overline{T})$, where \overline{T} is an average historic global temperature. Parameters for the GEV and bGEV – $\mu_0, \mu_t, \sigma, \xi$ – were estimated for each of the 5400 cases by maximum likelihood. For the bGEV, the beta distribution shape parameters α, β were kept at 5. For the cases where the GEV fit had negative ξ , bGEV was employed with different values of the quantile *a* ranging from 0.975 to 0.75, with *b* set to a - 0.01. If the fitted ξ was positive, a = 0.05, b = 0.2 were retained from [7].

Results showed that for both GEV and bGEV (regardless of a, b), most (91%) of the fitted values of ξ were negative, consistent with previous analyses of heat extremes [11, 12], with a median of about -0.22. The median μ_t was around 1.4, meaning that annual maximum temperatures warmed some 40% faster than global mean temperature. For GEV fits, the summed forecast NLL was infinite, because in a few cases (33 out of 5400; example shown in Figure 1) the forecast-year temperature hit a new record that was outside the support of the fitted GEV with negative ξ . The bGEV eliminated this problem, so the log likelihoods were all finite regardless of the chosen negative- ξ blending quantiles a, b. NLL summed across the 5400 forecasts reached a broad optimum for a between about 0.82 and 0.90 (Figure 2). Setting b to a - 0.05 or a - 0.15, instead of a - 0.01, was also tried and gave similar but slightly worse (higher) summed NLL. Overall, these initial results suggest that forecast quality using the bGEV distribution is not very sensitive to the exact choice of blending quantiles.

3. Discussion and conclusion

Many directions could be explored for improving forecasts of extreme events. Partial pooling of information on the bGEV distribution parameters across grid cells using hierarchical Bayesian modeling [19, 20] could enable more precise inference, as could the inclusion of climate covariates besides global mean temperature [21]. Such forecasts could also be used for other variables such as humid heat [22] and sea level extremes, and for compound hazards characterized by the simultaneous occurrence of multiple extremes, such as heat and drought or flood [23, 24]. A Bayesian approach also allows to include uncertainty due to bGEV parameter fitting in the forecast probability distribution [25]. Other extensions of the GEV distribution have also been



FIGURE 1. Example of a new temperature record from near Adelaide, Australia in December 2019, showing the histogram of the 1940-2018 annual maximum temperatures and the forecast for 2019 based on the GEV and bGEV. Both the GEV and bGEV appropriately shift the forecast probability distribution upward relative to the historic one dues to global warming, but the GEV distribution, with negative shape parameter, has zero probability mass past 312.3 K, whereas the bGEV forecast has positive, though small, probability for the actual 312.6 K (vertical line).

proposed and can be compared with the bGEV distribution, such as transmuted GEV [26] and odd GEV [27] distributions.

In summary, extending the bGEV distribution to cases with negative shape parameter allows better probabilistic forecasts of phenomena such as heat extremes by eliminating the sharp upper bound that is intrinsic to the GEV distribution with negative shape parameter but is not realistic for finite sample lengths.

Octave [28] programs to compute quantities related to the extended bGEV distribution and to fit the bGEV to the example data described above are available at https://github.com/nir-krakauer/bgev_octave. These build on functionality for working with the GEV distribution previously implemented in the Octave Statistics package [29].

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FIGURE 2. Summed negative log likelihood (NLL) across 5400 annual maximum temperature forecasts using the bGEV with different blending quantile hyperparameter values *a*. Lower NLL characterizes a better probabilistic forecast.

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